Fractal analysis of spiral trajectories of some planar vector fields, II

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Contents:

Introduction

D. Žubrinić, V. Županović, Fractal analysis of spiral trajectories of some planar vector fields, Bulletin des Sciences Mathématiques, 129/6 (2005), 457-485.

X planar vector field, $X(p) = 0$, p singularity

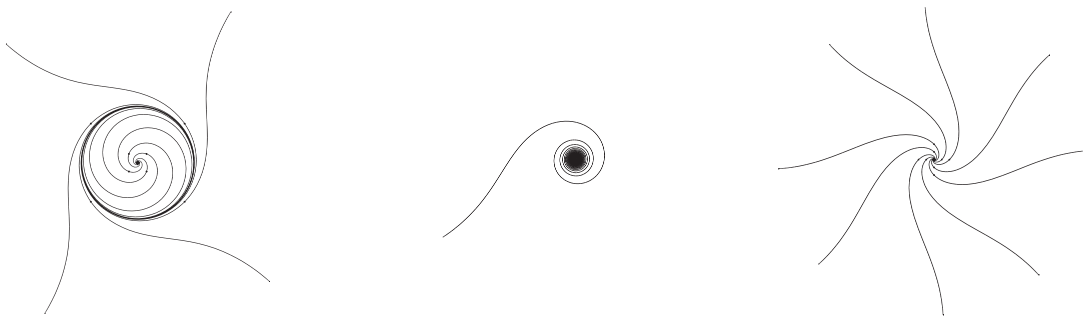

Figure 1: Limit cycle, weak focus, and strong focus

Strong focus: the matrix of the linear part of the vector field has both real and imaginary parts of eigenvalues nonzero.

Weak focus: both eigenvalues are pure imaginary and nonzero.

Usual question:

Can I take a simpler vector field and preserve topological type locally near the singularity?

Answer:

Yes, take a normal form!

The simplest case:

hyperbolic vector field, that is, a vector field with no eigenvalues on the imaginary axes.

The normal form is its linear part.

Other cases:

normal forms depend on the linear part.

Hopf-Takens bifurcation

Hopf bifurcation

Standard model where Hopf bifurcation occurs

$$\begin{aligned} X &= (-y + a_0 x + x(x^2+y^2))\frac{\partial}{\partial x} + \\ &\quad (x + a_0 y + y(x^2+y^2))\frac{\partial}{\partial y} \end{aligned}$$

$$\begin{cases} \dot{x} &= -y + a_0 x + x(x^2+y^2), \\ \dot{y} &= x + a_0 y + y(x^2+y^2). \end{cases} \tag{1}$$

In polar coordinates

$$\begin{cases} \dot{r} &= r(r^2 + a_0), \\ \dot{\varphi} &= 1. \end{cases}$$

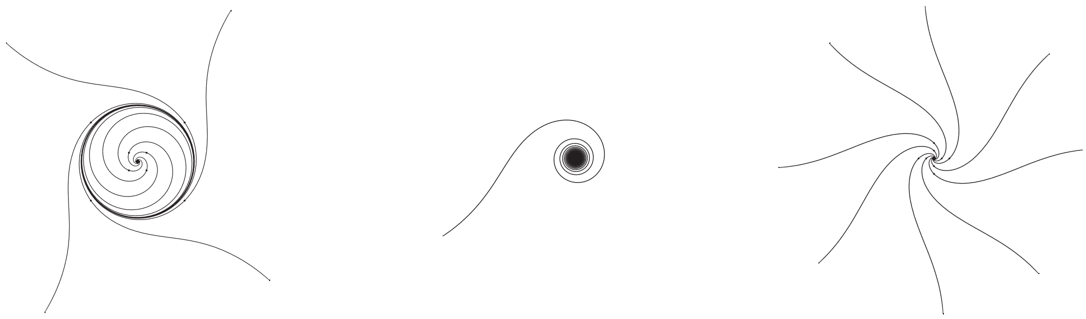

Figure 2: $a_0 < 0$, $a_0 = 0$, $a_0 > 0$

Hopf-Takens bifurcation

X $r-$parameter family of planar vector fields, $X(0) = 0$.

For $a_0 = ... = a_{r-1} = 0$ X has eigenvalues on the imaginary axes and nonzero.

Takens F., Unfoldings of certain singularities of vector fields: Generalized Hopf bifurcations, J. Differential Equations, 14 (1973) 476–493

X has the same phase portraits and bifurcations as $X_+^{(l)}$ or $X_-^{(l)}$, where

$$X_{\pm}^{(l)} = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) \pm \left((x^2+y^2)^l + \right.$$

$$\left. a_{l-1}(x^2+y^2)^{l-1} + \ldots + a_0\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right),$$

$(a_0, \ldots, a_{l-1}) \in \mathbb{R}^l$, is a normal form of X.

Standard generalized Hopf bifurcation or standard Hopf-Takens bifurcation

In polar coordinates

$$\begin{cases} \dot{r} &= r(r^{2l} + \sum_{i=0}^{l-1} a_i r^{2i}), \\ \dot{\varphi} &= 1. \end{cases} \qquad (2)$$

The case $l = 2$

$$\begin{cases} \dot{r} &= r(r^4 + a_1 r^2 + a_0), \\ \dot{\varphi} &= 1. \end{cases}$$

We take the fixed value $a_1 = -2$ and consider a_0 as a bifurcation parameter.

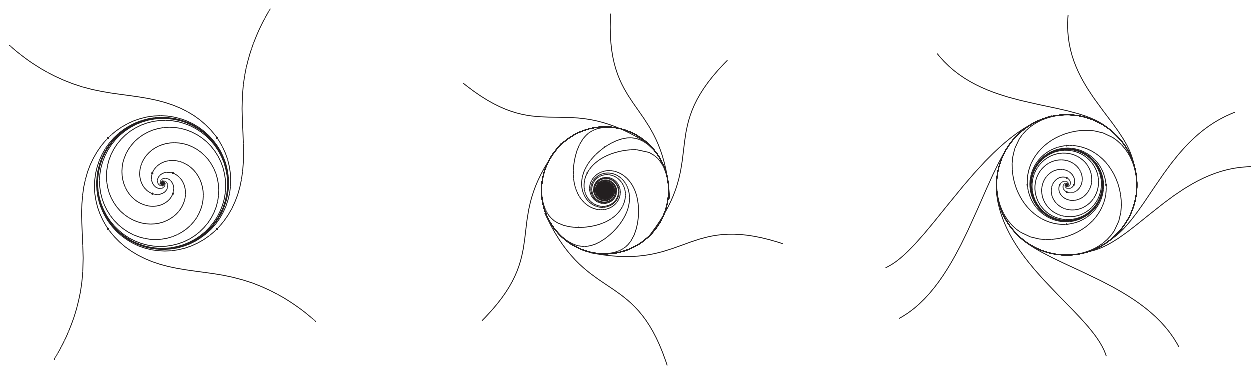

Figure 3: $a_0 < 0$, $a_0 = 0$, $a_0 \in (0, 1)$

Figure 4: $a_0 = 1$, $a_0 > 1$

Fractal analysis of Hopf-Takens bifurcation

Spiral $r = f(\varphi)$ of focus type is comparable with the spiral $r = \varphi^{-\alpha}$ of power type if

$$\underline{C}\varphi^{-\alpha} \leq f(\varphi) \leq \overline{C}\varphi^{-\alpha}$$

for some $\underline{C}, \overline{C} > 0$, and for all $\varphi \in [\varphi_1, \infty)$.

Analogously for spirals with negative orientation, that is, $\underline{C}|\varphi|^{-\alpha} \leq f(\varphi) \leq \overline{C}|\varphi|^{-\alpha}$ for $\varphi \in (-\infty, \varphi_1]$.

Spiral $r = f(\varphi)$ of focus type is comparable with the exponential spiral $r = e^{-\beta\varphi}$ if

$$\underline{C}e^{-\beta\varphi} \leq f(\varphi) \leq \overline{C}e^{-\beta\varphi}$$

for some $\underline{C}, \overline{C} > 0$ and $\beta > 0$, and for all $\varphi \in [\varphi_1, \infty)$.

Analogously for spirals with negative orientation, that is, for $\varphi \in (-\infty, \varphi_1]$ and $\beta < 0$.

Theorem 1 (The case of focus)

Γ *a part of a trajectory of (2) near the origin.*

(a) $a_0 \neq 0$*, then the spiral* Γ *is of exponential type, that is, comparable with* $r = e^{a_0\varphi}$*, and hence*

$$\dim_B \Gamma = \dim_B(\Gamma, rad) = 1.$$

(b) k *is fixed,* $1 \leq k \leq l$*,* $a_l = 1$ *and* $a_0 = \ldots = a_{k-1} = 0$*,* $a_k \neq 0$*. Then* Γ *is comparable with the spiral* $r = \varphi^{-1/2k}$*, and*

$$d := \dim_B \Gamma = \dim_B(\Gamma, rad) = \frac{4k}{2k+1}.$$

Γ *is Minkowski measurable in the classical and radial sense and* $\mathcal{M}^d(\Gamma) = \mathcal{M}^d(\Gamma, rad)$ *with the common value equal to explicit constant.*

Theorem 2 (The case of limit cycle) *Let the system (2) have limit cycle* $r = a$ *of multiplicity* m*,* $1 \leq m \leq l$*;* Γ_1 *and* Γ_2 *the parts of two trajectories of (2) near the limit cycle from outside and inside respectively.*

(a) Then Γ_1 *and* Γ_2 *are comparable with exponential spirals* $r = a \pm e^{-\beta\varphi}$ *when* $m = 1$*,* $\beta \neq 0$ *(depending only on coefficients* a_i*,* $0 \leq i \leq l - 1$*);*

(b) Γ_1 *and* Γ_2 *are comparable with power spirals* $r = a \pm \varphi^{-1/(m-1)}$ *when* $m > 1$*.*

In both cases we have

$$d := \dim_B \Gamma_i = \dim_B(\Gamma_i, rad) = 2 - \frac{1}{m}, \quad i = 1, 2.$$

For $m = 1$ *we have* $\mathcal{M}^d(h, \Gamma_i) = \mathcal{M}^d(h, \Gamma_i, rad) = 2/\beta$*,* $i = 1, 2$*, where* $h(\varepsilon) := \varepsilon(\log(1/\varepsilon))^{-1}$*.*

For $m > 1$ *the spirals are Minkowski measurable both in the classical and radial sense and* $\mathcal{M}^d(\Gamma_i) = \mathcal{M}^d(\Gamma_i, rad)$*,* $i = 1, 2$*.*

Example $l = 2$

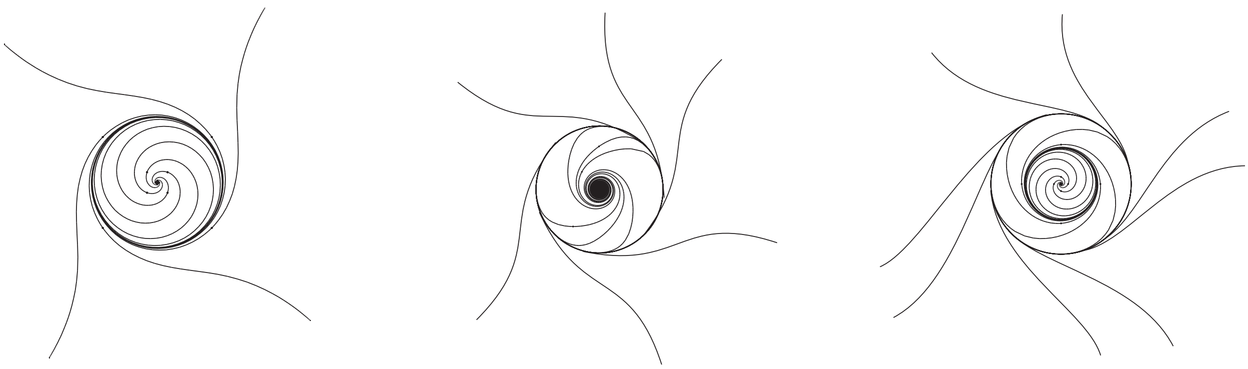

Figure 5: $a_0 < 0$, $a_0 = 0$, $a_0 \in (0, 1)$

Figure 6: $a_0 = 1$, $a_0 > 1$

(1) $a_0 < 0$ all box dimensions equal to 1 (trajectories of exponential type)

(2) $a_0 = 0$ $\dim_B \Gamma_1 = 4/3$, power case, here Γ_1 is a part of any trajectory near the origin.

Part near the limit cycle $r = \sqrt{2}$ has box dimension equal to 1 (exponential case).

(3) $a_0 \in (0, 1)$ we have two limit cycles of multiplicity one, and all box dimensions are equal to 1 (exponential case).

(4) $a_0 = 1$ we have limit cycle $r = 1$ of multiplicity two, and all trajectories near the limit cycle (either inside or outside) have box dimensions equal to $3/2$ (power case).

Trajectories inside the limit cycle, but near the origin, have box dimension equal to 1 (exponential case).

(5) $a_0 > 1$ box dimensions of all trajectories are equal to 1 (exponential case).

An idea

Here we deal only with systems with explicit solutions.

It is possible to control the “density” of a spiral by computing its box dimension even without solving a given system explicitely. For this the Poincaré map $P(x)$ (first return map) and displacement function $V(x) := x - P(x)$ give us information about such “density” of a spiral.

Announcement of results

(1) D. Žubrinić, V. Županović, Box dimension of spiral trajectories of some vector fields in $\mathbb{R}^3$, preprint.

Class of systems such that the linear part has a pure imaginary pair and a simple zero eigenvalues.

We found a class of systems in $\mathbb{R}^3$ such that the box dimension of spiral trajectories depends in nontrivial way on the coefficients of the system.

Notice that in the previous theorems box dimension depends only on the exponents.

All spirals in $\mathbb{R}^3$ are assumed to be contained in a two-dimensional surface.

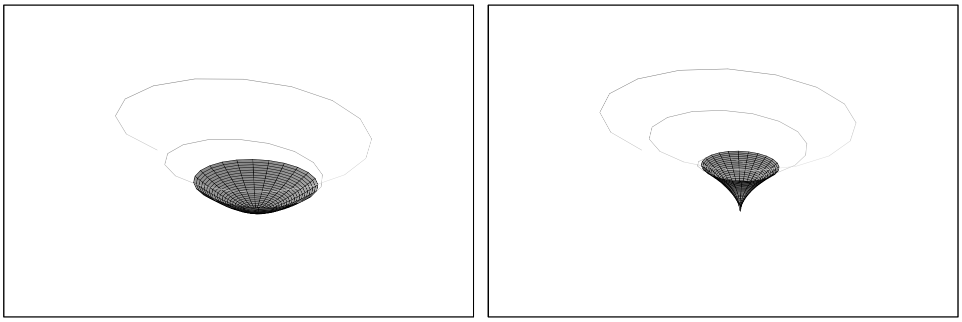

Figure 7: Focus spirals of Lipschitz and Hölder types

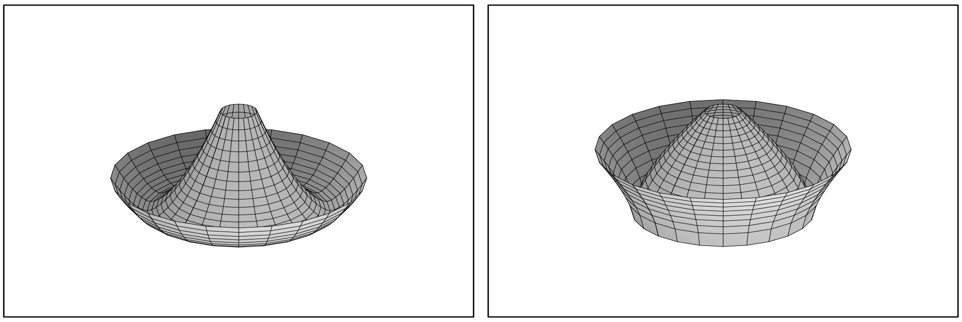

Figure 8: Surfaces containing cycle spirals of Lipschitz and Hölder types

Four types of spirals in $\mathbb{R}^3$:

- Lipschitz-focus spirals,
- Lipschitz-cycle spirals,
- Hölder-focus spirals,
- Hölder-cycle spirals.

Basic tool: box dimension and the nondegeneracy of a set is not affected by bi-Lipschitz mappings.

Only one of two projections of the spiral Γ onto horizontal and vertical planes, has box dimension equal to $\dim_B \Gamma$.

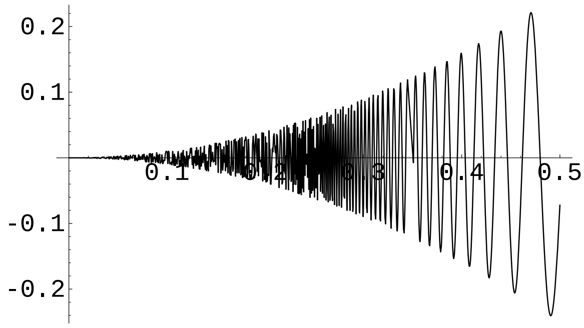

Figure 9: $(2, 4)$-chirp

Hölder-focus spiral,

$$r = \varphi^{-\alpha}, z = r^{\beta}$$

$\alpha \in (0, 1)$, $\beta \in (0, 1)$, $\varphi \in [\varphi_1, \infty)$.

Projection onto (y, z)-plane is

$$y = z^{1/\beta} \sin(z^{-1/\alpha\beta}).$$

Box dimension of graphs of such function (C. Tricot) is equal to

$$2 - \frac{\alpha(1+\beta)}{1+\alpha\beta}.$$

Such functions, called chirps, are also treated in

Pašić M., Županović V., Some metric-singular properties of the graph of solutions of the one-dimensional p-Laplacian, Electronic J. of Differential Equations, 60, 2004(2004), 1–25.

but in the different situation.

(2) N. Elezović, V. Županović, D. Žubrinić, Box dimension of trajectories of some discrete dynamical systems, work in progress

We investigate bifurcations of discrete dynamical systems. In one-dimensional case we consider what happens with box dimension and the Minkowski content when the saddle-node and the period doubling bifurcations occur. Example- Feigenbaum map.

Fractal dimensions in dynamics

We consider box dimension, there are many other names for box dimension (usually for the upper box dimension) appearing in the literature:

the Cantor-Minkowski order, Minkowski dimension, Bouligand dimension, Borel logarithmic rarefaction, Besicovitch-Taylor index, entropy dimension, Kolmogorov dimension, fractal dimension, and limit capacity.

Other fractal dimensions important for dynamics:

Hausdorff dimension, Ljapunov dimension, Rényi spectrum for dimensions, correlation dimension, information dimension, Hentschel-Procaccia spectrum for dimensions, packing dimension, and effective fractal dimension.

Since 1970 thermodynamics formalism, developed by Sinai, Ruelle, and Bowen, resulted in Hausdorff dimension of the Smale horseshoe and a lots of results about Hausdorff dimension of Julia and Mandelbrot sets.

Since 1980 physicists started to estimate and compute fractal dimensions of strange attractors (Lorenz, Henon,...). Fractal dimensions are estimated also for attractors of infinite-dimensional dynamical systems.

Our approach is different, we compute box dimension of a trajectory with respect to bifurcation. There are some well known bifurcations which we can explain in the new way through fractal analysis.

We have done it for Hopf-Takens bifurcation and now we work with saddle-node and period doubling bifurcations of discrete one-dimensional systems.