Fractal analysis of spiral trajectories of some planar vector fields

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Motivation

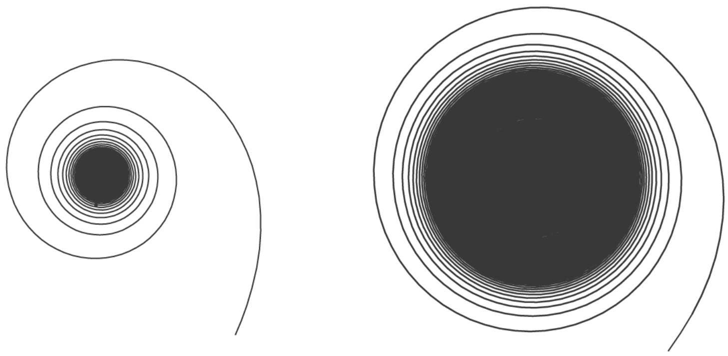

Figure 1: $r = \varphi^{-1/2}$, $r = \varphi^{-1/6}$

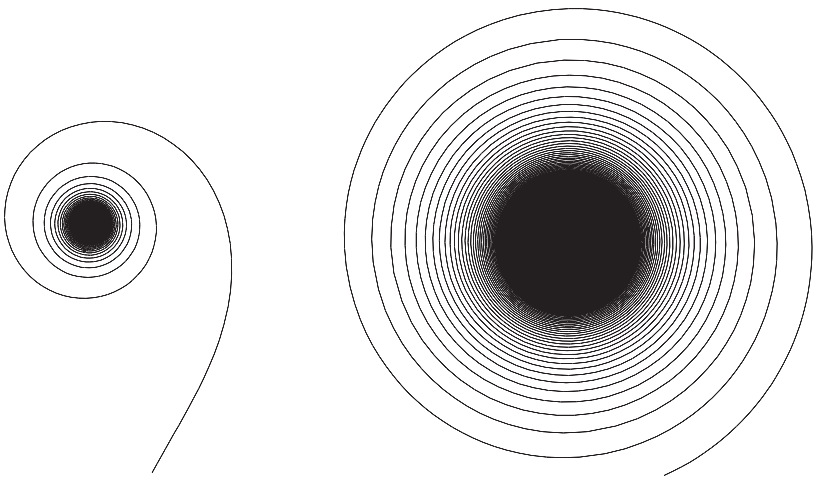

Figure 2: $r = \varphi^{-1/2}$, $r = 5\varphi^{-1/2}$

$$\begin{cases} \dot{r} &= r(r^4 - 2r^2 + a_0), \\ \dot{\varphi} &= 1. \end{cases}$$

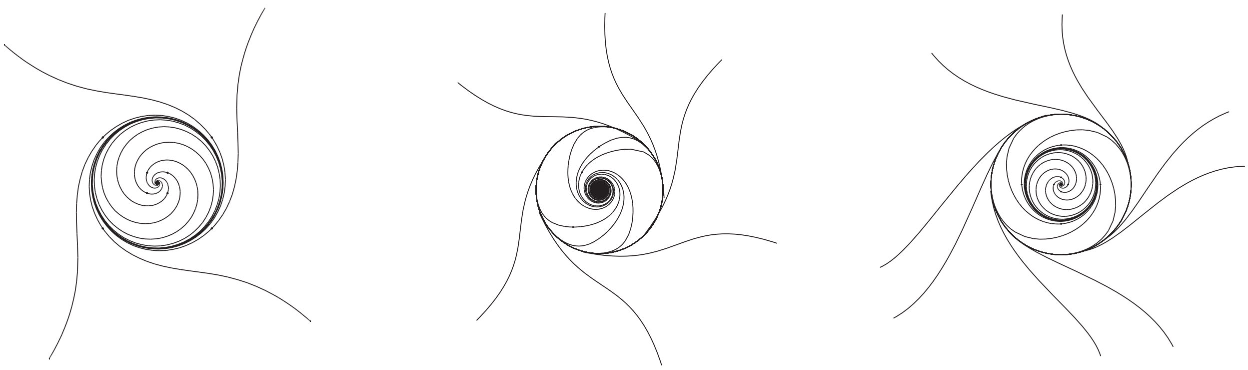

Figure 3: $a_0 < 0$, $a_0 = 0$, $a_0 \in (0, 1)$

Figure 4: $a_0 = 1$, $a_0 > 1$

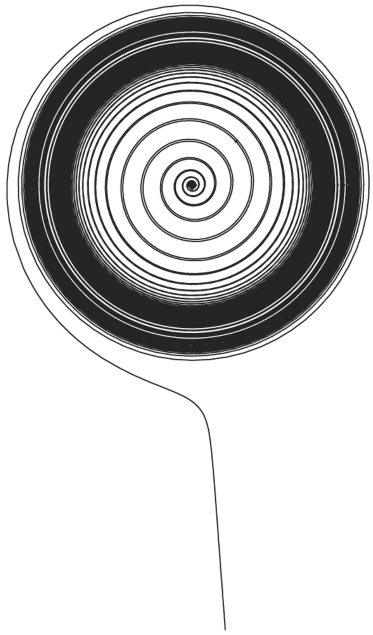

Figure 5: $\dot{r} = r(r - \frac{3}{4})^4(r + \frac{3}{4})^4$, $\dot{\varphi} = 1$.

D. Žubrinić, V. Županović: Fractal analysis of spiral trajectories of some planar vector fields, Bulletin des Sciences Mathématiques, 129/6 (2005), 457-485.

Box dimension and Minkowski content

$A \subset \mathbb{R}^N$ bounded,

Minkowski sausage of radius ε around A:
ε-neighbourhood of A,
$A_\varepsilon := \{y \in \mathbb{R}^N : d(y, A) < \varepsilon\}$.

Radial Minkowski sausage $A_{\varepsilon,rad}$ around A defined in the same way using radial distance function $d_{rad}(x, A)$ (as Euclidean distance from x to the set $A \cap \{tx : t \geq 0\}$, provided the intersection is nonempty, and ∞ otherwise).

Lower s-dimensional Minkowski content of A, $s \geq 0$:

$$\mathcal{M}_*^s(A) := \liminf_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^{N-s}},$$

analogously for the *upper s-dimensional Minkowski content* $\mathcal{M}^{*s}(A)$.

$$\underline{\dim}_B A := \inf\{s \geq 0 : \mathcal{M}_*^s(A) = 0\},$$
$$\overline{\dim}_B A := \inf\{s \geq 0 : \mathcal{M}^{*s}(A) = 0\}.$$

If $\mathcal{M}_*^s(A) = \mathcal{M}^{*s}(A)$ for some s, the common value is called *s-dimensional Minkowski content* of A, and is denoted by $\mathcal{M}^s(A)$.

If for some $d \geq 0$ we have that $\mathcal{M}^d(A) \in (0, \infty)$, then we say that the set A is *Minkowski measurable* (then clearly $d = \dim_B A$).

Radial s-dimensional Minkowski contents:

$$\begin{aligned} \mathcal{M}_*^s(A, rad) &:= \liminf_{\varepsilon \to 0} \frac{|A_{\varepsilon, rad}|}{\varepsilon^{N-s}}, \\ \mathcal{M}^{*s}(A, rad) &:= \limsup_{\varepsilon \to 0} \frac{|A_{\varepsilon, rad}|}{\varepsilon^{N-s}}. \end{aligned}$$

If for some $s \geq 0$ we have that $\mathcal{M}^s(A, rad) \in (0, \infty)$, then we say that the set A is *radially Minkowski measurable*.

In the case when lower or upper d-dimensional Minkowski contents of A are degenerate (0 or ∞), where $d = \dim_B A$, we deal with generalized Minkowski contents (C. He and M. Lapidus).

Find explicit positive *gauge functions*

$h_i : [0, \varepsilon_0) \to \mathbb{R}$, $i = 1, 2$, nondecreasing and $h_i(0) = 0$, such that the corresponding *generalized Minkowski contents*

$$\begin{aligned} \mathcal{M}_*(h_1, A) &:= \liminf_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^N} \cdot h_1(\varepsilon), \\ \mathcal{M}^*(h_2, A) &:= \limsup_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^N} \cdot h_2(\varepsilon), \end{aligned}$$

are nondegenerate. Similarly for radial generalized Minkowski contents

$$\mathcal{M}_*(h_1, A, rad) := \liminf_{\varepsilon \to 0} \frac{|A_{\varepsilon,rad}|}{\varepsilon^N} \cdot h_1(\varepsilon),$$

and $\mathcal{M}^*(h_2, A, rad)$.

Example - set A is a segment of length l

$$|A_\varepsilon| = 2l\varepsilon + \varepsilon^2 \pi$$

$$\mathcal{M}^s(A) = \lim_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^{2-s}} = \lim_{\varepsilon \to 0} (2l\varepsilon^{s-1} + \pi\varepsilon^s)$$

$$s = 1 \quad \mathcal{M}^1(A) = 2l$$

$$s < 1 \quad \mathcal{M}^s(A) = \infty$$

$$s > 1 \quad \mathcal{M}^s(A) = 0$$

Classification of spirals

We classify spirals in several different ways:

(a) spirals of *focus type* and of *limit cycle type*;

$r = f(\varphi)$, $f(\varphi) \to 0$ focus spiral

$r = 1 - \varphi^{-\alpha}$ limit cycle spiral

(b) spirals of *power, exponential, and logarithmic types*;

$r = \varphi^{-\alpha}$, $r = e^{-c\varphi}$, $r = (\log \varphi)^{-1}$

$r = 1 - \varphi^{-\alpha}$, $r = 1 - e^{-c\varphi}$, $r = 1 - (\log \varphi)^{-1}$

(c) spirals with *nondegenerate and degenerate* Minkowski contents.

$r = \varphi^{-1}$, $r = 1 - e^{-c\varphi}$

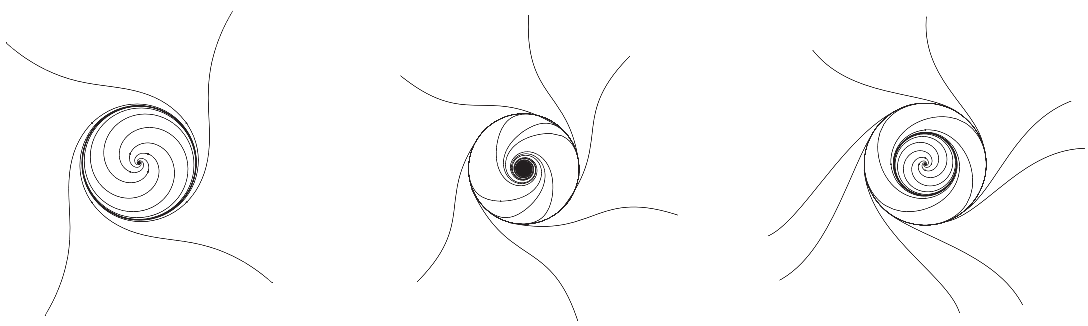

Figure 6: Spirals of focus and limit cycle types

Box dimension and Minkowski content of spirals

Spiral of focus type (that is, tending to the origin): the graph Γ of a function $r = f(\varphi)$, $\varphi \geq \varphi_1$ s.t.

$$\begin{cases} f : [\varphi_1, \infty) \to (0, \infty),\ f(\varphi) \to 0 \text{ as } \varphi \to \infty, \\ f \textit{ radially decreasing}, \text{ ie. for any } \varphi \geq \varphi_1, \\ \mathbb{N} \ni k \mapsto f(\varphi + 2k\pi) \text{ is decreasing} \end{cases}$$

Generalization of C. Tricot formula for box dimension of $r = \varphi^{-\alpha}$, $\alpha \in (0, 1)$

Theorem 1 (Power spirals of focus type)
Assumptions: $f : [\varphi_1, \infty) \to (0, \infty)$ *measurable, radially decreasing, such that for some positive* $\underline{m}$ *and* $\overline{m}$

$$\underline{m}\,\varphi^{-\alpha} \leq f(\varphi) \leq \overline{m}\,\varphi^{-\alpha}$$

for all $\varphi \geq \varphi_1 > 0$*; there exist positive* $\underline{a}$ *and* $\overline{a}$ *such that for all* $\varphi \geq \varphi_1$,

$$\underline{a}\,\varphi^{-\alpha-1} \leq f(\varphi) - f(\varphi + 2\pi) \leq \overline{a}\,\varphi^{-\alpha-1}.$$

By Γ *denote the graph* $r = f(\varphi)$*;* $\alpha \in (0, 1)$.

Conclusion:

$$d := \dim_B(\Gamma, rad) = \tfrac{2}{1+\alpha},$$
$$\underline{M} \le \mathcal{M}_*^d(\Gamma, rad) \le \mathcal{M}^{*d}(\Gamma, rad) \le \overline{M}$$

where $\underline{M}$ *and* $\overline{M}$ *do not depend on initial angle* φ_1*:*

$$\underline{M} \quad := \quad \pi \underline{m}^2 \left(\frac{2}{\overline{a}}\right)^{2\alpha/(1+\alpha)} + \frac{2\underline{m}}{1-\alpha}\left(\frac{\underline{a}}{2}\right)^{\frac{1-\alpha}{1+\alpha}}.$$

$$\overline{M} \quad := \quad \pi \overline{m}^2 \left(\frac{2}{\underline{a}}\right)^{2\alpha/(1+\alpha)} + \frac{2\overline{m}}{1-\alpha}\left(\frac{\overline{a}}{2}\right)^{\frac{1-\alpha}{1+\alpha}}.$$

Theorem 2 (Power spirals of limit cycle type)
Assumptions: $f : [\varphi_1, \infty) \to (0, \infty)$ *as in the preceding theorem,* α *a positive real number;* Γ *a spiral of the limit cycle type defined by* $r = 1 - f(\varphi)$*. Then*

$$d := \dim_B(\Gamma, rad) = \tfrac{2+\alpha}{1+\alpha},$$
$$\underline{M} \le \mathcal{M}_*^d(\Gamma, rad) \le \mathcal{M}^{*d}(\Gamma, rad) \le \overline{M},$$

where $\underline{M}$ *and* $\overline{M}$ *do not depend on initial angle* φ_1*:*

$$\underline{M} \quad := \quad 2\pi \underline{m} \left(\frac{2}{\overline{a}}\right)^{\alpha/(1+\alpha)} + 2\left(\frac{\underline{a}}{2}\right)^{1/(1+\alpha)},$$

$$\overline{M} \quad := \quad 2\pi\overline{m}\left(\frac{2}{\underline{a}}\right)^{\alpha/(1+\alpha)} + 2\left(\frac{\overline{a}}{2}\right)^{1/(1+\alpha)} .$$

Corollary 1 (Comparison with spirals of power type) *Assumptions:* $f : [\varphi_1, \infty) \to (0, \infty)$ *absolutely continuous,* $f(\varphi) \to 0$ *as* $\varphi \to \infty$*; and*

$$\underline{b}\,\varphi^{-\alpha-1} \le |f'(\varphi)| \le \overline{b}\,\varphi^{-\alpha-1}, \quad \varphi \ge \varphi_1, \quad (1)$$

where α, $\underline{b}$ *and* $\overline{b}$ *are positive constants.*

(a) Let Γ *be spiral defined by* $r = f(\varphi)$ *with* $\alpha \in (0, 1)$. *Then* $d := \dim_B(\Gamma, rad) = \frac{2}{1+\alpha}$, *and radial* d*-dimensional Minkowski contents of* Γ *are nondegenerate.*

(b) Let Γ *be a spiral* $r = 1 - f(\varphi)$ *with* $\alpha > 0$. *Then* $d := \dim_B(\Gamma, rad) = \frac{2+\alpha}{1+\alpha}$, *and radial* d*-dimensional Minkowski contents of* Γ *are nondegenerate.*

(c) In particular, conclusions in (a) and (b) hold if f *is of class* C^1 *and there exists*

$$\lim_{\varphi\to\infty} \frac{f'(\varphi)}{(\varphi^{-\alpha})'} \in (0, \infty). \quad (2)$$

Excision property of Minkowski content of spirals

Lemma 1 (Excision property for simple smooth curves) *Let* Γ *be a simple smooth curve in* $\mathbb{R}^2$*, that is,* Γ *is the graph of continuously differentiable injection* $h : [\varphi_1, \infty) \to \mathbb{R}^2$*. Assume that* $\underline{\dim}_B \Gamma > 1$*. Let* $\overline{\varphi}_1 > \varphi_1$ *be given and* $\Gamma_1 := h(\overline{\varphi}_1, \infty)$*. Then*

$$\underline{d} := \underline{\dim}_B \Gamma_1 = \underline{\dim}_B \Gamma,$$
$$\overline{d} := \overline{\dim}_B \Gamma_1 = \overline{\dim}_B \Gamma$$
$$\mathcal{M}_*^{\underline{d}}(\Gamma_1) = \mathcal{M}_*^{\underline{d}}(\Gamma), \quad \mathcal{M}^{*\overline{d}}(\Gamma_1) = \mathcal{M}^{*\overline{d}}(\Gamma).$$

Analogous claim holds for radial box dimensions and radial Minkowski contents.

Theorem 3 (Minkowski measurable spirals) *Assumptions:* $f : [\varphi_1, \infty) \to (0, \infty)$ *decreasing,* C^2 *function and* $\varphi_1 > 0$*; for every* $\varphi_0 > \varphi_1$ *there exist positive numbers* $\underline{m}(\varphi_0)$*,* $\overline{m}(\varphi_0)$*,* $D_1(\varphi_0)$ *and* $D_2(\varphi_0)$ *such that for all* $\varphi \geq \varphi_0$*,*

$$\underline{m}(\varphi_0)\, \varphi^{-\alpha} \leq f(\varphi) \leq \overline{m}(\varphi_0)\, \varphi^{-\alpha}$$
$$D_1(\varphi_0)\, \varphi^{-\alpha-1} \leq |f'(\varphi)| \leq D_2(\varphi_0)\, \varphi^{-\alpha-1},$$

and there exists a positive constant D_3 such that $|f''(\varphi)| \leq D_3\varphi^{-\alpha}$*; assume that*

$$\lim_{\varphi\to\infty} \underline{m}(\varphi) = \lim_{\varphi\to\infty} \overline{m}(\varphi) =: m,$$
$$\lim_{\varphi\to\infty} D_1(\varphi) = \lim_{\varphi\to\infty} D_2(\varphi) =: D.$$

Let Γ *be either the spiral* $r = f(\varphi)$ *of focus type (here we assume that* $\alpha \in (0, 1)$ *and define* $d := 2/(1 + \alpha)$*), or the spiral* $r = 1 - f(\varphi)$ *of limit cycle type (here we assume that* $\alpha > 0$ *and define* $d := (2 + \alpha)/(1 + \alpha)$*). Then* $\dim_B \Gamma = \dim_B(\Gamma, rad) = d$*, the spirals are Minkowski measurable both in the classical and radial sense, and moreover,*

$$\mathcal{M}^d(\Gamma) = \mathcal{M}^d(\Gamma, rad)$$
$$= \begin{cases} \pi m^2(\pi D)^{-2\alpha/(1+\alpha)} + \frac{2m}{1-\alpha}(\pi D)^{-\frac{1-\alpha}{1+\alpha}} \\ 2\pi(\pi D)^{-\alpha/(1+\alpha)} + 2(\pi D)^{1/(1+\alpha)} \end{cases}$$

respectively.

Fractal analysis of Hopf-Takens bifurcation

Hopf-Takens bifurcation

Standard generalized Hopf bifurcation or standard Hopf-Takens bifurcation

In polar coordinates

$$\left\{ \begin{array}{lcl} \dot{r} & = & r(r^{2l} + \sum_{i=0}^{l-1} a_i r^{2i}), \\ \dot{\varphi} & = & 1. \end{array} \right. \tag{3}$$

Spiral $r = f(\varphi)$ of focus type is *comparable with the spiral $r = \varphi^{-\alpha}$ of power type* if

$$\underline{C}\varphi^{-\alpha} \leq f(\varphi) \leq \overline{C}\varphi^{-\alpha}$$

for some $\underline{C}, \overline{C} > 0$, and for all $\varphi \in [\varphi_1, \infty)$.

Analogously for spirals with negative orientation, that is, $\underline{C}|\varphi|^{-\alpha} \leq f(\varphi) \leq \overline{C}|\varphi|^{-\alpha}$ for $\varphi \in (-\infty, \varphi_1]$.

Spiral $r = f(\varphi)$ of focus type is *comparable with the exponential spiral $r = e^{-\beta\varphi}$* if

$$\underline{C}e^{-\beta\varphi} \leq f(\varphi) \leq \overline{C}e^{-\beta\varphi}$$

for some $\underline{C}, \overline{C} > 0$ and $\beta > 0$, and for all $\varphi \in [\varphi_1, \infty)$.

Analogously for spirals with negative orientation, that is, for $\varphi \in (-\infty, \varphi_1]$ and $\beta < 0$.

Theorem 4 (The case of focus)

Γ a part of a trajectory of (3) near the origin.

(a) $a_0 \neq 0$, then the spiral Γ is of exponential type, that is, comparable with $r = e^{a_0\varphi}$, and hence

$$\dim_B \Gamma = \dim_B(\Gamma, rad) = 1.$$

(b) k is fixed, $1 \leq k \leq l$, $a_l = 1$ and $a_0 = \ldots = a_{k-1} = 0$, $a_k \neq 0$. Then Γ is comparable with the spiral $r = \varphi^{-1/2k}$, and

$$d := \dim_B \Gamma = \dim_B(\Gamma, rad) = \frac{4k}{2k+1}.$$

Γ is Minkowski measurable in the classical and radial sense and $\mathcal{M}^d(\Gamma) = \mathcal{M}^d(\Gamma, rad)$ with the common value equal to explicit constant.

Theorem 5 (The case of limit cycle) *Let the system (3) have limit cycle* $r = a$ *of multiplicity* m, $1 \le m \le l$; Γ_1 *and* Γ_2 *the parts of two trajectories of (3) near the limit cycle from outside and inside respectively.*

(a) Then Γ_1 *and* Γ_2 *are comparable with exponential spirals* $r = a \pm e^{-\beta\varphi}$ *when* $m = 1$, $\beta \neq 0$ *(depending only on coefficients* a_i, $0 \le i \le l-1$*);*

(b) Γ_1 *and* Γ_2 *are comparable with power spirals* $r = a \pm \varphi^{-1/(m-1)}$ *when* $m > 1$.

In both cases we have

$$d := \dim_B \Gamma_i = \dim_B(\Gamma_i, rad) = 2 - \frac{1}{m}, \quad i = 1, 2.$$

For $m = 1$ *we have* $\mathcal{M}^d(h, \Gamma_i) = \mathcal{M}^d(h, \Gamma_i, rad) = 2/\beta$, $i = 1, 2$, *where* $h(\varepsilon) := \varepsilon(\log(1/\varepsilon))^{-1}$.

For $m > 1$ *the spirals are Minkowski measurable both in the classical and radial sense and* $\mathcal{M}^d(\Gamma_i) = \mathcal{M}^d(\Gamma_i, rad)$, $i = 1, 2$.

Example $l = 2, a_1 = -2$

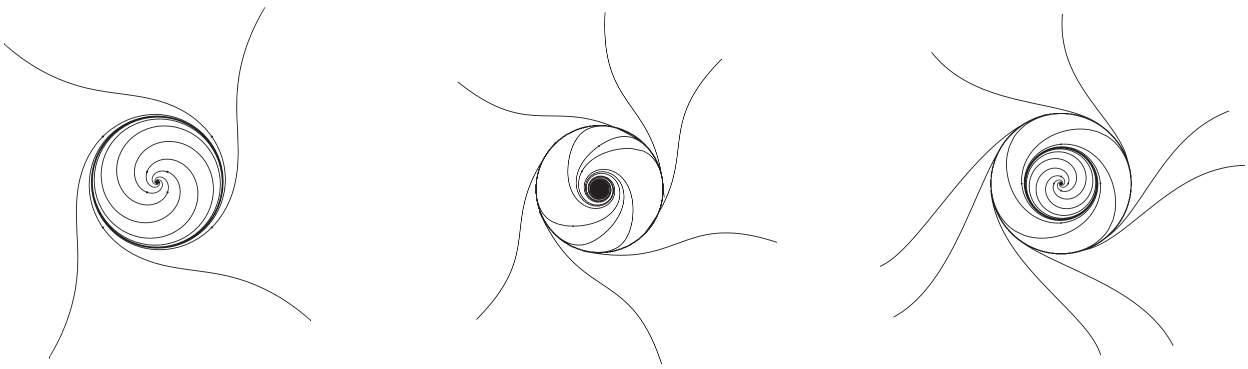

Figure 7: $a_0 < 0$, $a_0 = 0$, $a_0 \in (0, 1)$

Figure 8: $a_0 = 1$, $a_0 > 1$

(1) $a_0 < 0$ all box dimensions equal to 1 (trajectories of exponential type)

(2) $a_0 = 0$ $\dim_B \Gamma_1 = 4/3$, power case, here Γ_1 is a part of any trajectory near the origin.

Part near the limit cycle $r = \sqrt{2}$ has box dimension equal to 1 (exponential case).

(3) $a_0 \in (0, 1)$ we have two limit cycles of multiplicity one, and all box dimensions are equal to 1 (exponential case).

(4) $a_0 = 1$ we have limit cycle $r = 1$ of multiplicity two, and all trajectories near the limit cycle (either inside or outside) have box dimensions equal to $3/2$ (power case).

Trajectories inside the limit cycle, but near the origin, have box dimension equal to 1 (exponential case).

(5) $a_0 > 1$ box dimensions of all trajectories are equal to 1 (exponential case).

Continuation of the work

Here we deal only with systems with explicit solutions.

It is possible to control the "density" of a spiral by computing its box dimension even without solving a given system explicitely. For this the Poincaré map $P(x)$ and displacement function $V(x) := x - P(x)$ give us information about such "density" of a spiral.

N. Elezović, V. Županović, D. Žubrinić: Box dimension of trajectories of some discrete dynamical systems, to appear in Chaos, Solitons and Fractals

We investigate bifurcations of discrete one-dimensional dynamical systems. We consider what happens with box dimension when the saddle-node and the period doubling bifurcations occur. We noticed the same phenomenon as in this case, box dimension jumps before peridic orbit is born.

D. Žubrinić, V. Županović: Box dimension of spiral trajectories of some vector fields in $\mathbb{R}^3$, preprint

Class of systems such that the linear part has a pure imaginary pair and a simple zero eigenvalues.

$$\begin{aligned} \dot{r} &= c_1 r^3 + \ldots + c_m r^{2m+1} \\ \dot{\varphi} &= 1 \\ \dot{z} &= d_2 z^2 + \ldots + d_n z^n \end{aligned} \tag{4}$$

We found a class of systems in $\mathbb{R}^3$ such that the box dimension of spiral trajectories Γ depends in nontrivial way on the coefficients of the system, an example is

$$\dot{r} = a_1 r z, \quad \dot{\varphi} = 1, \quad \dot{z} = b_2 z^2 \tag{5}$$

$$\dim_B \Gamma = \frac{2}{1 + a_1/b_2}, \quad \frac{a_1}{b_2} \in (0, 1] \tag{6}$$

Basic tool: box dimension and the nondegeneracy of a set is not affected by bi-Lipschitz mappings.

Only one of two projections of the spiral Γ onto horizontal and vertical planes, has box dimension equal to $\dim_B \Gamma$.

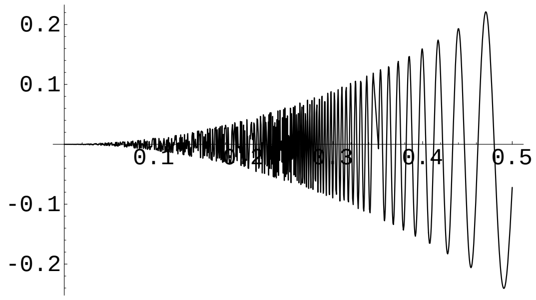

Figure 9: $(2, 4)$-chirp, $\alpha = \beta = 2$

Projection of

$$r = \varphi^{-\alpha}, z = r^{\beta}$$

$\alpha \in (0, 1)$, $\beta \in (0, 1)$, $\varphi \in [\varphi_1, \infty)$.

onto (y, z)-plane is

$$y = z^{1/\beta} \sin(z^{-1/\alpha\beta}).$$

Box dimension of graphs of such function (C. Tricot) is equal to $2 - \frac{\alpha(1+\beta)}{1+\alpha\beta}$.