BOX DIMENSION OF TRAJECTORIES OF SOME DISCRETE DYNAMICAL SYSTEMS

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Abstract. We study the asymptotics, box dimension, and Minkowski content of trajectories of some discrete dynamical systems. We show that under very general conditions, trajectories corresponding to parameters where saddle-node bifurcation appears have box dimension equal to 1/2, while those corresponding to period-doubling bifurcation parameter have box dimension equal to 2/3. Furthermore, all these trajectories are Minkowski nondegenerate. The results are illustrated in the case of logistic map.

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1. Introduction

We are interested in bifurcation parameters μ of discrete one-dimensional dynamical systems in the sense of nontriviality of box dimension of the trajectory S_μ, near a given trajectory of the system. More precisely, we are interested in values of the parameter μ such that $\dim_B S_\mu$ is nonzero. The main results are stated in Theorems 7 and 8.

A typical example is the system generated by standard logistic map. M. Feigenbaum studied the dynamics of the logistic map for $\lambda \in (0, 4]$. Taking $\lambda = \lambda_\infty \approx 3.570$ the corresponding invariant set $A \subset [0, 1]$ has both Hausdorff and box dimensions equal to ≈ 0.538 (Grassberger [3], Grassberger and Procaccia [4]). Here we compute precise values of box dimension of trajectories corresponding to period-doubling bifurcation parameters 3 and $1 + \sqrt{6}$, and to period-3 bifurcation parameter $1 + \sqrt{8}$, see Corollary 1.

Similar effect of nontriviality of box dimension of trajectories as in bifurcation problems for discrete systems has been noticed for some planar vector fields having spiral trajectories Γ_μ, see [11]. There we have considered a standard model of Hopf-Takens bifurcation with respect to bifurcation parameter μ where $\dim_B \Gamma_\mu > 1$, while $\dim_B \Gamma_\mu = 1$ otherwise. We noticed

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that a limit cycle is born at the moment of jump of box dimension of a spiral trajectory. Analogously, in the case of one-dimensional discrete system a periodic trajectory is born at the moment of jump of box dimension of a discrete trajectory (sequence). We expect that fractal analysis of general planar spiral trajectories can be reduced to the study of discrete one-dimensional trajectories via the Poincaré map, see also [11, Remark 11]. A review of results dealing with applications of fractal dimensions to dynamics is given in [12].

We recall the notions of box dimension and Minkowski content, see e.g. Mattila [6]. For any subset $S \subset \mathbb{R}^N$ by S_ε we denote the ε-neighbourhood of S (also called Minkowski sausage of radius ε around A, a term coined by B. Mandelbrot), and $|S_\varepsilon|$ is its N-dimensional Lebesgue measure. For a bounded set S and given $s \ge 0$ we define the upper s-dimensional Minkowski content of S by

$$\mathcal{M}^{*s}(S) = \limsup_{\varepsilon \to 0} \frac{|S_\varepsilon|}{\varepsilon^{N-s}}.$$

Analogously for the lower s-dimensional content of S. The upper box dimension of S is defined by

$$\overline{\dim}_B S = \inf\{s \ge 0 : \mathcal{M}^{*s}(S) = 0\},$$

and analogously the lower box dimension $\underline{\dim}_B S$. If both dimensions coincide, we denote it by $\dim_B S$. We say that a set S is Minkowski nondegenerate if its d-dimensional upper and lower Minkowski contents are in $(0, \infty)$ for some $d \ge 0$, and Minkowski measurable if $\mathcal{M}^{*d}(S) = \mathcal{M}^d_*(S) := \mathcal{M}^d(S) \in (0, \infty)$.

Nondegeneracy of Minkowski contents for fractal strings has been characterized by Lapidus and van Frankenhuysen [5]. Applications of Minkowski content in the study of singular integrals can be seen in [9] and [10].

For any two sequences $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ of positive real numbers we write $a_n \simeq b_n$ as $n \to \infty$ if there exist positive constants A and B such that $A \le a_n/b_n \le B$ for all n. Analogously, for two positive functions $f, g : (0, r) \to \mathbb{R}$ we write $f(x) \simeq g(x)$ as $x \to 0$ if $f(x)/g(x) \in [A, B]$ for x sufficiently small.

2. Box dimension of some recurrently defined sequences

The first result deals with sequences $(x_n)_{n\ge1}$ converging monotonically to zero.

Theorem 1. *Let $\alpha > 1$ and let $f : (0, r) \to (0, \infty)$ be a monotonically nondecreasing function such that $f(x) \simeq x^\alpha$ as $x \to 0$, and $f(x) < x$ for all $x \in (0, r)$. Consider the sequence $S(x_1) := (x_n)_{n\ge1}$ defined by*

$$x_{n+1} = x_n - f(x_n), \quad x_1 \in (0, r). \tag{1}$$

Then

$$x_n \simeq n^{-1/(\alpha-1)} \quad \textit{as } n \to \infty. \tag{2}$$

Furthermore,

$$\dim_B S(x_1) = 1 - \frac{1}{\alpha}, \tag{3}$$

and the set $S(x_1)$ *is Minkowski nondegenerate.*

Proof. (a) Assuming that $Ax^\alpha \leq f(x) \leq Bx^\alpha$, we have

$$0 < x_{n+1} \leq x_n - Ax_n^\alpha. \tag{4}$$

It is easy to see that x_n tends monotonically to 0. Using induction we first prove that $x_n \leq bn^{-\beta}$, where $\beta := \frac{1}{\alpha-1}$, for some positive constant b. Let us consider inductive step first, and then the basis. Assume that $x_n \leq bn^{-\beta}$ for some n, and assume also that $x_n \leq x_{max}$, where x_{max} is the point of maximum of $x - x^\alpha$, $x > 0$. Note that since x_n is decreasing, converging to zero, then $x_n \leq x_{max}$ for all n sufficiently large. Exploiting monotonicity of $x \mapsto x - x^\alpha$ on $(0, x_{max})$ we have

$$x_{n+1} \leq x_n - Ax_n^\alpha \leq bn^{-\beta} - Ab^\alpha n^{-\alpha\beta} \leq b(n+1)^{-\beta}.$$

In order to prove the last inequality, it suffices to show that

$$n^{-\beta} - Ab^{\alpha-1}n^{-\alpha\beta} \leq (n+1)^{-\beta}.$$

To this end let us consider the binomial series expansion:

$$\begin{aligned}(n+1)^{-\beta} &= [n(1+\frac{1}{n})]^{-\beta} = n^{-\beta} + \binom{-\beta}{1} n^{-\beta-1} + R_n \\ &= n^{-\beta} - \beta n^{-\alpha\beta} + R_n \geq n^{-\beta} - Ab^{\alpha-1}n^{-\alpha\beta}.\end{aligned} \tag{5}$$

The last inequality holds provided b is chosen so that $Ab^{\alpha-1} \geq \beta$, and if $R_n \geq 0$. To prove $R_n \geq 0$ note that $R_n = a_2 + a_4 + a_6 + \dots$, where each a_k (with even k) has the form

$$\begin{aligned}a_k &= \binom{-\beta}{k} n^{-\beta-k} + \binom{-\beta}{k+1} n^{-\beta-k-1} \\ &= \frac{\beta(\beta+1)\dots(\beta+k-1)}{k!} n^{-\beta-k} - \frac{\beta(\beta+1)\dots(\beta+k)}{(k+1)!} n^{-\beta-k-1}.\end{aligned}$$

Inequality $a_k \geq 0$ is equivalent with

$$n \geq \frac{\beta+k}{k+1} = \frac{1}{(k+1)(\alpha-1)} + \frac{k}{k+1}.$$

For all even k the right-hand side obviously does not exceed $n_0 = n_0(\alpha) = \frac{1}{3(\alpha-1)} + 1$. The condition $x_n \leq x_{max}$ for $n \geq n_0$ is secured if we take n_0 sufficiently large. From condition $Ab^{\alpha-1} \geq \beta$ we see that we must take $b \geq (\beta/A)^\beta$. Hence, the basis of induction and inductive step hold for $n \geq n_0$ with such a b. Taking b still larger, we can achieve that $x_n \leq bn^{-\beta}$ for all $n \geq 1$.

(b) To prove that there exists $a > 0$ such that $x_n \geq an^{-\beta}$ for all $n \geq 1$, we use only $x_{n+1} \geq x_n - Bx_n^\alpha$. Assuming by induction that the desired inequality holds for a fixed n we obtain analogously as in (a) that

$$x_{n+1} \geq x_n - Bx_n^\alpha \geq an^{-\beta} - Ba^\alpha n^{-\alpha\beta} \geq a(n+1)^{-\beta}, \tag{6}$$

under the assumption that $x_n \leq x_{max}$. In order to show the last inequality in (6) we use binomial expansion (5) again, and proceed by writing $\beta = \gamma + \delta$ with arbitrarily chosen positive constants γ and δ. We have to achieve

$$\begin{aligned}(n+1)^{-\beta} &= (n^{-\beta} - \gamma n^{-\alpha\beta}) - (\delta n^{-\alpha\beta} - R_n) \\ &\leq n^{-\beta} - Ba^{\alpha-1} n^{-\alpha\beta}.\end{aligned}$$

This holds provided $\gamma \geq Ba^{\alpha-1}$, that is, $a \leq (\gamma/B)^\beta$, and if $R_n \leq \delta n^{-\alpha\beta}$ for some $\delta \in (0, \beta)$. Note that $R_n \leq \delta n^{-\alpha\beta}$ is equivalent with

$$n \sum_{k=2}^{\infty} \binom{\beta}{k} n^{-k} \leq \delta.$$

that is, with

$$n \left[(1 + \frac{1}{n})^\beta - 1 - \binom{\beta}{1} \frac{1}{n} \right] \leq \delta.$$

Using Taylor's formula $(1 + \frac{1}{n})^\beta = 1 + \binom{\beta}{1}\frac{1}{n} + \binom{\beta}{2}\overline{x}^2$, $0 < \overline{x} < \frac{1}{n}$, we see that the above inequality is satisfied when $\binom{\beta}{2} n\overline{x}^2 \leq \delta$, that is, when $\binom{\beta}{2} n^{-1} \leq \delta$. This holds for all $n \geq n_0$ if n_0 is large enough. We can choose n_0 large enough so that also $x_{n_0} \leq x_{max}$. Taking a small enough we can achieve the basis of induction, $x_{n_0} \geq an_0^{-\beta}$. Taking $a > 0$ still smaller the lower bound will hold for all $n \geq 1$. This completes the proof of the lower bound of x_n by induction.

(c) Since f is nondecreasing, the sequence $l_n := x_n - x_{n+1} = f(x_n)$ is nonincreasing. Hence, we can derive Minkowski nondegeneracy of $S(x_1)$ using Lapidus and Pomerance [5, Theorem 2.4]. Indeed, from (2) we have

$$l_n = f(x_n) \simeq x_n^\alpha \simeq n^{-\alpha/(\alpha-1)} = n^{-1/d},$$

where $d := 1 - \frac{1}{\alpha} \in (0, 1)$. Using the mentioned result we conclude that $S(x_1)$ is Minkowski nondegenerate and $\dim_B S(x_1) = d = 1 - \frac{1}{\alpha}$. □

Remark 1. Step (c) in the proof of Theorem 1 can be carried out by directly estimating Minkowski contents of $S = S(x_1)$. Using $l_n := x_n - x_{n+1} = f(x_n) \leq B(bn^{-\beta})^\alpha = Bb^\alpha n^{-\alpha\beta}$ we see that $l_n \leq 2\varepsilon$ if $n \geq (\frac{1}{2}Bb^\alpha)^d \varepsilon^{-d}$, where $d := 1 - \frac{1}{\alpha}$. Defining $n_0 = n_0(\varepsilon) := \lceil (\frac{1}{2}Bb^\alpha)^d \varepsilon^{-d} \rceil$ we have

$$|S_\varepsilon| \geq x_{n_0} + 2\varepsilon(n_1 - 1), \tag{7}$$

where $n_1 = n_1(\varepsilon)$ is obtained in the similar way from the condition $l_n = f(x_n) \geq 2\varepsilon$. It is satisfied for $n \leq n_1 := \lfloor (\frac{1}{2}Aa^\alpha)^d \varepsilon^{-d} \rfloor$. Using (7) we conclude that

$$\mathcal{M}_*^d(S) \geq \frac{a}{b}\left(\frac{2}{B}\right)^{1/\alpha} + 2\left(\frac{1}{2}Aa^\alpha\right)^d. \tag{8}$$

In the analogous way, estimating $|S_\varepsilon|$ from above, we obtain

$$\mathcal{M}^{*d}(S) \leq \frac{b}{a}\left(\frac{2}{A}\right)^{1/\alpha} + 2\left(\frac{1}{2}Bb^\alpha\right)^d. \tag{9}$$

This proves that S is Minkowski nondegenerate and $\dim_B S = d$.

Remark 2. We do not know if the set $S = S(x_1)$ corresponding to $f(x) = A \cdot x^\alpha$ in Theorem 1, where $A > 0$, is Minkowski measurable. Numerical experiments show that in this case any corresponding sequence $S = (x_n)_{n\geq 1}$, $x_1 \in (0,1)$, is Minkowski measurable, and

$$\mathcal{M}^d(S) = \left(\frac{2}{A}\right)^{1/\alpha} \frac{\alpha}{\alpha - 1}. \tag{10}$$

This value is obtained if we let formally $a = b = (\beta/A)^\beta$ in (8) and (9).

The following result deals with sequences $(x_n)_{n\geq 1}$ oscillating around a fixed point x_0, so that their two subsequences defined by odd and even indices monotonically converge to x_0. It suffices to consider the case $x_0 = 0$.

Theorem 2. *Let $f : (-r, r) \to \mathbb{R}$ be a function such that $f(x) \simeq |x|^\alpha$ as $x \to 0$, where $\alpha > 1$. We also assume that the function $f(-x - f(x)) - f(x)$ satisfies the following conditions:*

(11) *it is monotonically nondecreasing for $x > 0$ small enough,*
it is monotonically nonincreasing for $x < 0$ small enough,

$$f(-x - f(x)) - f(x) \simeq \pm|x|^{2\alpha-1} \quad \text{as } x \to 0\pm. \tag{12}$$

Then there exists $r_1 > 0$ such that for any sequence $S(x_1) := (x_n)_{n\geq 1}$ defined by

$$x_{n+1} = -x_n - f(x_n), \quad x_1 \in (-r_1, r_1), \tag{13}$$

we have

$$|x_n| \simeq n^{-1/(2\alpha-2)}, \quad \text{as } n \to \infty. \tag{14}$$

Furthermore,

$$\dim_B S(x_1) = 1 - \frac{1}{2\alpha - 1}, \tag{15}$$

and the set $S(x_1)$ is Minkowski nondegenerate.

Proof. Note that if we define $F(x) := -x - f(x)$ then $F^2(x) = F(F(x)) = x - g(x)$, where $g(x) := f(-x - f(x)) - f(x)$. We have

$$g(x) \simeq \pm|x|^{2\alpha-1} \quad \text{as } x \to 0\pm. \tag{16}$$

As $2\alpha - 1 > 1$ and $g(0) = 0$, from (16) we see that there exists $r_1 \in (0, r)$ such that $0 < g(x) < x$ for $x \in (0, r_1)$ and $-x < g(x) < 0$ for $x \in (-r_1, 0)$.

Starting with $x_1 \in (0, r_1)$, the sequence $y_n := x_{2n-1}$, $n \geq 1$, satisfies $y_{n+1} = y_n - g(y_n)$, and since (y_n) is nonincreasing, it is contained in $(0, r_1)$. By Theorem 1 applied to g and the sequence (y_n) we have that $y_n = x_{2n-1} \simeq n^{-1/(2\alpha-2)}$ as $n \to \infty$. To obtain the same asymptotics for $|x_{2n}|$, it suffices to start with $x_2 = F(x_1) < 0$, and to consider the sequence $z_n := x_{2n}$, $n \geq 1$, contained in $(-r, 0)$. Using again Theorem 1 (modified to this situation; note that $-x < g(x) < 0$) applied to the sequence z_n, we obtain $|z_n| = |x_{2n}| \simeq n^{-1/(2\alpha-2)}$. Hence $|x_n| \simeq n^{-1/(2\alpha-2)}$. The same asymptotics is obtained if we start with $x_1 \in (-r_1, 0)$.

Exploiting finite stability of the upper box dimension, see Falconer [2, p. 44], we have that $\overline{\dim}_B S = \max\{\overline{\dim}_B S_-, \overline{\dim}_B S_+\}$, where $S_- := S \cap (-r, 0)$ and $S_+ := S \cap (0, r)$ are negative and positive part of the sequence $S = S(x_1)$. Since $\overline{\dim}_B S_- = \overline{\dim}_B S_+ = 1 - \frac{1}{2\alpha-1}$, see Theorem 1, we conclude that $\overline{\dim}_B S = 1 - \frac{1}{2\alpha-1}$.

To estimate the lower box dimension, first note that the sets S_+ and S_- are separated by $x = 0$, hence $(S_+)_\varepsilon \cap (S_-)_\varepsilon = (-\varepsilon, \varepsilon)$. Therefore $|S_\varepsilon| = |(S_+)_\varepsilon| + |(S_-)_\varepsilon| - |(S_+)_\varepsilon \cap (S_-)_\varepsilon| = |(S_+)_\varepsilon| + |(S_-)_\varepsilon| - 2\varepsilon$, and from this we immediately obtain that $\mathcal{M}^d_*(S) \geq \mathcal{M}^d_*(S_+) + \mathcal{M}^d_*(S_-) > 0$, where $d := 1 - \frac{1}{2\alpha-1}$. We have used also Minkowski nondegeneracy of $S_\pm$. Hence, $\underline{\dim}_B S \geq d$. This finishes the proof of $\dim_B S = d$. □

Remark 3. Conditions of Theorem 2 are satisfied when for example $f(x) = |x|^\alpha$, $x \in (-r, r)$.

In order to facilitate the study of bifurcation problems below, it will be convenient to formulate the following consequences of Theorems 1 and 2.

Theorem 3. *Let $F : (x_0 - r, x_0 + r) \to \mathbb{R}$ be a function of class C^3, such that*

$$F(x_0) = x_0, \tag{17}$$

$$F'(x_0) = 1, \tag{18}$$

$$F''(x_0) < 0. \tag{19}$$

Then there exists $r_1 > 0$ such that for any sequence $S(x_1) = (x_n)_{n\geq 1}$ defined by

$$x_{n+1} = F(x_n), \quad x_1 \in (x_0, x_0 + r_1),$$

we have $|x_n - x_0| \simeq n^{-1}$ as $n \to \infty$,

$$\dim_B S(x_1) = \frac{1}{2}, \tag{20}$$

and $S(x_1)$ is Minkowski nondegenerate. Analogous result holds if $x_1 \in (x_0 - r_1, x_0)$, assuming that in (19) we have the opposite sign.

Proof. We can assume without loss of generality that $x_0 = 0$, and let $x > 0$. By the Taylor formula, using (17), (18), we have that

$$F(x) = x - f(x),$$

where

$$f(x) = -\frac{F''(0)}{2!}x^2 - \frac{F'''(\overline{x})}{3!}x^3$$

with $\overline{x} = \overline{x}(x) \in (0, r)$. Since $F''(0) < 0$, we see that $f(x) \simeq x^2$. The condition $f(x) < x$ is clearly satisfied in $(0, r_1)$ for r_1 sufficiently small. The function f is increasing, since using again Taylor's formula applied to $F' \in C^2$, we get

$$f'(x) = 1 - F'(x) = -F''(0)\,x - \frac{F'''(\overline{x})}{2!}\,x^2 > 0$$

for $x \in (0, r_1)$ with r_1 small enough. The claim follows from Theorem 1 with $\alpha = 2$. Analogously for $x \in (-r_1, 0)$. □

Theorem 4. *Let $F : (x_0 - r, x_0 + r) \to \mathbb{R}$ be a function of class C^4, such that*

$$F(x_0) = x_0, \tag{21}$$

$$F'(x_0) = -1, \tag{22}$$

$$F''(x_0) \neq 0. \tag{23}$$

Then there exists $r_1 > 0$ such that for any sequence $S(x_1) = (x_n)_{n\geq 1}$ defined by

$$x_{n+1} = F(x_n), \quad x_1 \in (x_0 - r_1, x_0 + r_1), \quad x_1 \neq x_0,$$

we have $|x_n - x_0| \simeq n^{-1/2}$ as $n \to \infty$,

$$\dim_B S(x_1) = \frac{2}{3}, \tag{24}$$

and $S(x_1)$ is Minkowski nondegenerate.

Proof. We assume without loss of generality that $x_0 = 0$. It suffices to check that conditions of Theorem 2 are satisfied. Using Taylor's formula, (17), and (23), we get

$$F(x) = -x - f(x),$$

where

$$f(x) = -\frac{F''(0)}{2!}\,x^2 - \frac{F'''(\overline{x})}{3!}\,x^3,$$

where $\overline{x} = \overline{x}(x) \in (-r, r)$. Now we consider the function $g(x) = f(-x - f(x)) - f(x)$:

$$g(x) = -F''(0)x \cdot f(x) + \cdots = \frac{1}{2}F''(0)^2x^3 + \text{higher order terms}.$$

Since $g'(x) = \frac{3}{2}F''(0)^2\,x^2 + \text{higher order terms} > 0$ for $x \neq 0$ such that $|x|$ is small enough, it is clear that $g(x)$ is increasing on $(-r_1, r_1)$, provided $r_1 > 0$ is small enough. Also,

$$g(x) = \frac{F''(0)^2}{2}x^3 + o(x^3) \simeq x^3 \quad \text{as } x \to 0.$$

This shows that conditions (11) and (12) are fulfilled. The claim follows from Theorem 2 with $\alpha = 2$. □

Remark 4. It is clear that more general versions of Theorems 3 and 4 can be proved where finitely many consecutive derivatives of F of orders $k = 2, 3, \dots, 2m-1$ are equal to zero, and $F^{(2m)}(x_0) \neq 0$.

Lemma 1. *Assume that $S = (x_n) \subset \mathbb{R}$ is a sequence of positive numbers converging exponentially to zero, that is, there exists $\lambda \in (0,1)$ and a constant $C > 0$ such that $0 < x_n \le C\lambda^n$ for all n. Then $\dim_B S = 0$.*

Proof. For any fixed $\varepsilon > 0$ inequality $x_n \le \lambda^n < 2\varepsilon$ is satisfied for $n \ge n_0(\varepsilon) := \lceil \frac{\log(2\varepsilon)}{\log\lambda} \rceil$. Hence,

$$|S_\varepsilon| \le 2\varepsilon + n_0(\varepsilon)\cdot 2\varepsilon,$$

and from this $\mathcal{M}^{*s}(S) = 0$ for any $s \in (0,1)$. Therefore, $\overline{\dim}_B S = 0$. □

From this we immediately obtain the following result. The notion of hyperbolic fixed point of the system is described e.g. in Devaney [1].

Theorem 5. (Hyperbolic fixed point) *Let $F : (x_0 - r, x_0 + r) \to \mathbb{R}$ be a function of class C^1, $F(x_0) = x_0$, and $|F'(x_0)| < 1$. Then there exists $r_1 \in (0, r)$ such that for any sequence $S(x_1) = (x_n)_{n\ge1}$ defined by*

$$x_{n+1} = F(x_n), \quad x_1 \in (x_0 - r_1, x_0 + r_1)$$

we have

$$\dim_B S(x_1) = 0.$$

It is easy to see that under the assumptions on f given in Theorem 1 we have that for each $\lambda \in (0,1)$ the sequence $S := (x_n)_{n\ge1}$ corresponding to $x_{n+1} = \lambda x_n - g(x_n)$, $x_1 \in (0, r_\lambda)$, with r_λ sufficiently small, has exponential decay, $0 < x_n \le \lambda^n$. Hence $\dim_B S = 0$.

Now we state a simple but useful comparison result, which we shall need in the proof of Theorem 6.

Lemma 2. (Comparison principle for box dimensions) *Assume that $A = (a_n)_{n\ge1}$ and $B = (b_n)_{n\ge1}$ are two decreasing sequences of positive real numbers converging to zero, such that the sequences of their differences $(a_n - a_{n+1})_{n\ge1}$ and $(b_n - b_{n+1})_{n\ge1}$ are monotonically nonincreasing. If $a_n \le b_n$ then*

$$\overline{\dim}_B A \le \overline{\dim}_B B, \quad \underline{\dim}_B A \le \underline{\dim}_B B.$$

Proof. Using the fact that the Borel rarefaction index of A is equal to the upper box dimension, see Tricot [8, p. 34 and Theorem on p. 35], we obtain

$$\overline{\dim}_B A = \varlimsup_{n\to\infty} \frac{1}{1 + \frac{\log(1/a_n)}{\log n}}. \tag{25}$$

Since $0 < a_n \le b_n$ we conclude that $\overline{\dim}_B A \le \overline{\dim}_B B$. Using methods described in Tricot [8, pp. 33–36] it can be shown that analogous result holds for the lower box dimension:

$$\underline{\dim}_B A = \varliminf_{n\to\infty} \frac{1}{1 + \frac{\log(1/a_n)}{\log n}}. \tag{26}$$

This immediately implies $\underline{\dim}_B A \le \underline{\dim}_B B$. □

Now we consider a sequence with a very slow convergence to 0, such that its box dimension is maximal possible. We achieve this by assuming that $f(x)$ converges very fast to 0 as $x \to 0$. An example of such a function is $f(x) = \exp(-1/x)$.

Theorem 6. *Let $f : (0, r) \to (0, \infty)$ be a nondecreasing function such that $f(x) < x$ and for any $\alpha > 1$ we have that $f(x) = O(x^\alpha)$ as $x \to 0$. Consider the sequence $S := (x_n)_{n\ge1}$ defined by $x_{n+1} = x_n - f(x_n)$, $x_1 \in (0, r)$. Then*

$$\dim_B S = 1. \tag{27}$$

Proof. It is clear that $x_n \to 0$. For any fixed $\alpha > 1$ there exists $B_\alpha > 0$ such that we have $f(x) \le B_\alpha x^\alpha$, hence $x_{n+1} \ge x_n - B_\alpha x_n^\alpha$ for all $n \ge 1$. As in step (b) of the proof of Theorem 1 we conclude that there exists $a = a_\alpha > 0$ such that $x_n \ge an^{-1/(\alpha-1)}$ for all n. Since $x_n \to 0$ monotonically, then $c_n = x_n - x_{n+1} = f(x_n) \to 0$ also monotonically. Therefore, using Lemma 2 (see also (26)), we get

$$\underline{\dim}_B S \ge \underline{\dim}_B \{an^{-1/(\alpha-1)}\} = \frac{1}{1 + (\alpha - 1)^{-1}} = 1 - \frac{1}{\alpha}.$$

The claim follows by letting $\alpha \to \infty$. □

3. Box dimension of trajectories at bifurcation points

For definitions of saddle-node (or tangent) bifurcation and period-doubling bifurcation and basic results see Devaney [1, Section 1.12]. Note that conditions in Theorem 7 are essentially the same as those in [1, Theorem 12.6]. Also conditions in Theorem 8 are essentially the same as those in [1, Theorem 12.7]. The novelty in Theorems 7 and 8 are precise values of box dimensions of trajectories near the point of bifurcation, convergence rate of trajectories, and their Minkowski nondegeneracy.

Theorem 7. (Saddle-node bifurcation) *Suppose that a function $F : J \times (x_0 - r, x_0 + r) \to \mathbb{R}$, where J is an open interval in $\mathbb{R}$, is such that $F(\lambda_0, \cdot)$ is of class C^3 for some $\lambda_0 \in \mathbb{R}$, and $F(\cdot, x)$ of class C^1 for all x. Assume that*

$$F(\lambda_0, x_0) = x_0, \tag{28}$$

$$\frac{\partial F}{\partial x}(\lambda_0, x_0) = 1, \tag{29}$$

$$\frac{\partial^2 F}{\partial x^2}(\lambda_0, x_0) < 0, \tag{30}$$

$$\frac{\partial F}{\partial \lambda}(\lambda_0, x_0) \ne 0. \tag{31}$$

Then λ_0 is the point where saddle-node bifurcation occurs. Furthermore, there exists $r_1 \in (0, r)$ such that for any sequence $S(\lambda_0, x_1) = (x_n)_{n\ge1}$ defined *by*

$$x_{n+1} = F(\lambda_0, x_n), \quad x_1 \in (x_0, x_0 + r_1),$$

we have $|x_n - x_0| \simeq n^{-1}$ *as* $n \to \infty$,

$$\dim_B S(\lambda_0, x_1) = \frac{1}{2}, \tag{32}$$

and $S(\lambda_0, x_1)$ *is Minkowski nondegenerate. Analogous result holds if* $x_1 \in (x_0 - r_1, x_0)$, *assuming that in (30) we have the opposite sign.*

Proof. The claim follows from Theorem 3 and [1, Theorem 12.6]. □

Theorem 8. (Period-doubling bifurcation) *Let* $F : J \times (x_0 - r, x_0 + r) \to \mathbb{R}$ *be a function of class* C^2, *where* J *is an open interval in* $\mathbb{R}$, *and* $F(\lambda_0, \cdot)$ *is of class* C^4 *for some* $\lambda_0 \in J$. *Assume that*

$$F(\lambda_0, x_0) = x_0, \tag{33}$$

$$\frac{\partial F}{\partial x}(\lambda_0, x_0) = -1, \tag{34}$$

$$\frac{\partial^2 F}{\partial x^2}(\lambda_0, x_0) \neq 0, \tag{35}$$

$$\frac{\partial^2 (F^2)}{\partial \lambda\, \partial x}(\lambda_0, x_0) \neq 0, \qquad \frac{\partial^3 (F^2)}{\partial x^3}(\lambda_0, x_0) \neq 0, \tag{36}$$

where we have denoted $F^2 = F \circ F$. *Then* λ_0 *is the point where period-doubling bifurcation occurs. Furthermore, there exists* $r_1 \in (0, r)$ *such that for any sequence* $S(\lambda_0, x_1) = (x_n)_{n \geq 1}$ *defined by*

$$x_{n+1} = F(\lambda_0, x_n), \quad x_1 \in (x_0 - r_1, x_0 + r_1), \quad x_1 \neq x_0,$$

we have $|x_n - x_0| \simeq n^{-1/2}$ *as* $n \to \infty$,

$$\dim_B S(\lambda_0, x_1) = \frac{2}{3}, \tag{37}$$

and $S(\lambda_0, x_1)$ *is Minkowski nondegenerate.*

Proof. The claim follows from Theorem 4 and [1, Theorem 12.7]. □

Now we apply preceding results to bifurcation problem generated by the logistic map. By $d(x, A)$, where $x \in \mathbb{R}$ and $A \subset \mathbb{R}$, we denote Euclidean distance from x to A.

Corollary 1. (Logistic map) *Let* $F(\lambda, x) = \lambda x(1 - x)$, $x \in (0, 1)$, *and let* $S(\lambda, x_1) = (x_n)_{n \geq 1}$ *be a sequence defined by initial value* x_1 *and* $x_{n+1} = F(\lambda, x_n)$.

(a) For $\lambda_0 = 1$, *taking* $x_1 > 0$ *sufficiently close to* $x_0 = 0$, *we have that* $x_n \simeq n^{-1}$ *as* $n \to \infty$, *and*

$$\dim_B S(1, x_1) = \frac{1}{2}.$$

(b) (Onset of period-2 cycle) For $\lambda_0 = 3$ the corresponding fixed point is $x_0 = 2/3$. For any x_1 sufficiently close to x_0 we have that $|x_n - x_0| \simeq n^{-1/2}$, and

$$\dim_B S(3, x_1) = \frac{2}{3}.$$

(c) For any $\lambda \notin \{1, 3\}$ and x_1 such that the sequence $S(\lambda, x_1)$ is convergent, we have that $\dim_B S(\lambda, x_1) = 0$.

(d) (Onset of period-4 cycle) If $\lambda_0 = 1 + \sqrt{6}$ then for any x_1 sufficiently close to period-2 trajectory $A = \{a_1, a_2\}$ we have that $d(x_n, A) \simeq n^{-1/2}$ as $n \to \infty$, and

$$\dim_B S(1 + \sqrt{6}, x_1) = \frac{2}{3}.$$

(e) (Period-3 cycle) Let $\lambda_0 = 1 + \sqrt{8}$ and let a_1, a_2, a_3 be fixed points of F^3 such that $0 < a_1 < a_2 < a_3 < 1$, $F(a_1) = a_2$, $F(a_2) = a_3$, and $F(a_3) = a_1$. Then there exists $\delta > 0$ such that for any initial value

$$x_1 \in (a_1 - \delta, a_1) \cup (a_2 - \delta, a_2) \cup (a_3, a_3 + \delta)$$

we have $d(x_n, \{a_1, a_2, a_3\}) \simeq n^{-1}$ as $n \to \infty$, and

$$\dim_B S(1 + \sqrt{8}, x_1) = \frac{1}{2}.$$

All trajectories appearing in this corollary are Minkowski nondegenerate.

Proof. Claim (a) follows from Theorem 1. For (b) and (c) see Theorems 8 and 5. Claim (d) follows from Theorem 4 since $(F^2)''(\lambda_0, x_0) \neq 0$.

Claim (e) follows using Theorem 3 applied to F^3. Note that these three intervals are disjoint for $\delta > 0$ small enough. See Strogatz [7, pp. 362 and 363]. The fact that for $\lambda_0 = 1 + \sqrt{8}$ we have $(F^3)''(a_i) \neq 0$, $i = 1, 2, 3$, can be obtained by direct computation. □

Remark 5. It would be interesting to know precise values of box dimensions of trajectories corresponding to all period-doubling bifurcation parameters λ_k where 2^k-periodic points occur. On the basis of numerical experiments we expect that all of them will be equal to 2/3.

Example. For $F(\lambda, x) := \lambda e^x$, see Devaney [1, Section 1.12], we can obtain similar results. Indeed, using Theorem 7 we obtain that

$$\dim_B S(e^{-1}, x_1) = \frac{1}{2}$$

for all x_1 in a punctured neighbourhood of $x_0 = 1$. Using Theorem 8 we obtain that for any x_1 in a punctured neighbourhood of $x_0 = -1$ we have

$$\dim_B(-e, x_1) = \frac{2}{3}.$$

If $\lambda \notin \{e^{-1}, -e\}$, we have $\dim_B S(\lambda, x_1) = 0$ provided $S(\lambda, x_1)$ is a convergent sequence, see Theorem 5.

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