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Fractal Dimensions in Dynamics

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Introduction

p0005 Since the 1970s, dimension theory for dynamics has evolved into an independent field of mathematics. Its main goal is to measure complexity of invariant sets and measures using fractal dimensions. The history of fractal dimensions is closely related to the names of H Minkowski (Minkowski content, 1903), H Hausdorff (Hausdorff dimension, 1919), G Bouligand (Bouligand dimension, 1928), LS Pontryagin and LG Schnirelmann (metric order, 1932), P Moran (Moran geometric constructions, 1946), AS Besicovitch and SJ Taylor (Besicovitch–Taylor index, 1954), A Rényi (Rényi spectrum for dimensions, 1957), AN Kolmogorov and VM Tihomirov (metric dimension, Kolmogorov complexity, 1959), YaG Sinai, D Ruelle, R Bowen (thermodynamic formalism, Bowen's equation, 1972, 1973, 1979), B Mandelbrot (fractals and multifractals, 1974), JL Kaplan and JA Yorke (Lyapunov dimension, 1979), JE Hutchinson (fractals and self-similarity, 1981), C Tricot, D Sullivan (packing dimension, 1982, 1984), HGE Hentschel and I Procaccia (Hentschel–Procaccia spectrum for dimensions, 1983), Ya Pesin (Carathéodory–Pesin dimension, 1988), M Lapidus and M van Frankenhuysen (complex dimensions for fractal strings, 2000), etc. Fractal dimensions enable us to have a better insight into the dynamics appearing in various problems in physics, engineering, chemistry, medicine, geology, meteorology, ecology, economics, computer science, image processing, and, of course, in many branches of mathematics. Concentrating on box and Hausdorff dimensions only, we describe basic methods of fractal analysis in dynamics, sketch their applications, and indicate some trends in this rapidly growing field.

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Fractal Dimensions

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Box Dimensions

p0010 Let A be a bounded set in $\mathbb{R}^N$, and let $d(x, A)$ be Euclidean distance from x to A. The Minkowski sausage of radius ε around A (a term coined by B Mandelbrot) is defined as ε-neighborhood of A, that is, $A_\varepsilon := \{y \in \mathbb{R}^N \colon d(y, A) < \varepsilon\}$. By the upper s-dimensional Minkowski content of A, $s \geq 0$, we mean

$$\mathcal{M}^{*s}(A) := \overline{\lim_{\varepsilon \to 0}} \frac{|A_\varepsilon|}{\varepsilon^{N-s}} \in [0, \infty]$$

Here $|\cdot|$ denotes N-dimensional Lebesgue measure. The corresponding upper box dimension is defined by

$$\overline{\dim}_{\mathrm{B}} A := \inf\{s \geq 0 \colon \mathcal{M}^{*s}(A) = 0\}$$

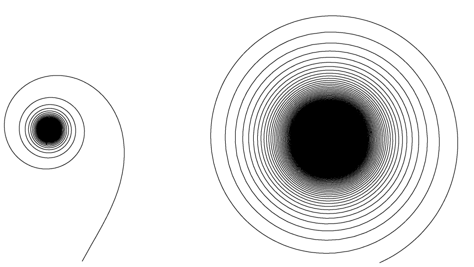
The lower s-dimensional Minkowski content $\mathcal{M}_*^s(A)$ and the corresponding lower box dimension $\underline{\dim}_{\mathrm{B}} A$ are defined analogously. The name of box dimension stems from the following: if we have an ε-grid in $\mathbb{R}^N$ composed of closed N-dimensional boxes with side ε, and if $N(A, \varepsilon)$ is the number of boxes of the grid intersecting A, then

$$\overline{\dim}_{\mathrm{B}} A = \overline{\lim_{\varepsilon \to 0}} \frac{\log N(A, \varepsilon)}{\log(1/\varepsilon)}$$

and analogously for $\underline{\dim}_{\mathrm{B}} A$. It suffices to take any geometric subsequence $\varepsilon_k = b^{-k}$ in the limit, where $b > 1$ (H. Furstenberg, 1970). AU3 There are many other names for the upper box dimension appearing in the literature, like the Cantor–Minkowski order, Minkowski dimension, Bouligand dimension, Borel logarithmic rarefaction, Besicovitch–Taylor index, entropy dimension, Kolmogorov dimension, fractal dimension, capacity dimension, and limit capacity. If A is such that $\underline{\dim}_{\mathrm{B}} A = \overline{\dim}_{\mathrm{B}} A$, the common value is denoted by $d := \dim_{\mathrm{B}} A$, and we call it the box dimension of A. If, in addition to this, both $\mathcal{M}_*^d(A)$ and $\mathcal{M}^{*d}(A)$ are in $(0, \infty)$, we say that A is Minkowski nondegenerate. If, moreover, $\mathcal{M}_*^d(A) = \mathcal{M}^{*d}(A) =: \mathcal{M}^d(A) \in (0, \infty)$, then A is said to be Minkowski measurable.

Assume that A is such that $d := \dim_{\mathrm{B}} A$ and $\mathcal{M}^d(A)$ exist. Then the value of $\mathcal{M}^d(A)^{-1}$ is called the lacunarity of A (B Mandelbrot, 1982). AU4 A bounded set $A \subset \mathbb{R}^N$ is said to be porous (A Denjoy, 1920) if there exist $\alpha > 0$ and $\delta > 0$ such that for every $x \in A$ and $r \in (0, \delta)$ there is $y \in \mathbb{R}^N$ such that the open ball $B_{\alpha r}(y)$ is contained in $B_r(x) \setminus A$. If A is porous then it is easy to see that $\overline{\dim}_{\mathrm{B}} A < N$ (O Martio and M Vuorinen, 1987, A Salli, 1991). p0015

We proceed with two examples. Let $A := C^{(a)}$, $a \in (0, 1/2)$, be the Cantor set obtained from $[0, 1]$ by consecutive deletion of 2^k middle open intervals of length $a^k(1 - 2a)$ in step $k \in \mathbb{N} \cup \{0\}$. Then $\dim_{\mathrm{B}} A = (\log 2)/(\log(1/a))$ (G Bouligand, 1928), and A is nondegenerate, but not Minkowski measurable (Lapidus and Pomerance, 1993). For the spiral Γ of focus type defined by $r = m\varphi^{-\alpha}$ in polar coordinates, where $\alpha \in (0, 1)$ and $m > 0$ are p0020



f0005 **Figure 1** Spirals of equal box dimensions (4/3) and different lacunarities (0.43 and 0.05).

fixed, $\varphi \geq \varphi_1 > 0$, we have $\dim_B \Gamma = 2/(1+\alpha)$ (M Mendés-France, Y Dupain, C Tricot, 1983). It is Minkowski measurable (Žubrinić and Županović, 2005), and the larger m, the smaller the lacunarity; see **Figure 1**.

s0020 Hausdorff Dimension

p0025 For a given subset A of $\mathbb{R}^N$ (not necessarily bounded) and $s \geq 0$ we define $\mathcal{H}^s(A) := \lim_{\varepsilon \to 0} \inf\{\sum_{i=1}^{\infty} r_i{}^s\} \in [0, \infty]$, where the infimum is taken over all finite or countable coverings of A by open balls of radii $r_i \leq \varepsilon$. The value of $\mathcal{H}^s(A)$ is called s-dimensional Hausdorff outer measure of A. The Hausdorff dimension of A, sometimes called the Hausdorff–Besicovitch dimension, is defined by

$$\dim_{\mathrm{H}} A := \inf\{s \geq 0\colon\ \mathcal{H}^s(A) = 0\}$$

If A is bounded then $\dim_{\mathrm{H}} A \leq \underline{\dim}_{\mathrm{B}} A \leq \overline{\dim}_{\mathrm{B}} A \leq N$.

p0030 We say that A is Hausdorff nondegenerate (or d-set) if $\mathcal{H}^d(A) \in (0, \infty)$ for some $d \geq 0$. Cantor sets share this property, and $\dim_{\mathrm{H}} C^{(a)} = (\log 2)/(\log(1/a))$, where $a \in (0, 1/2)$ (Hausdorff, 1919).

s0025 Gauge Functions

p0035 The notions of Minkowski contents and Hausdorff measure can be generalized using gauge functions $h\colon [0, \varepsilon_0) \to \mathbb{R}$ that are assumed to be continuous, increasing, and $h(0) = 0$. For example,

$$\mathcal{M}^{*h}(A) := \overline{\lim_{\varepsilon \to 0}} \frac{|A_\varepsilon|}{\varepsilon^N} h(\varepsilon)$$

and similarly for $\mathcal{M}^h_*(A)$ (M Lapidus and C He, 1997), while for $\mathcal{H}^h(A)$ it suffices to change $r_i{}^s$ with $h(r_i)$ in the above definition of the Hausdorff outer measure (Besicovitch, 1934). Gauge functions are used for sets that are Minkowski or Hausdorff degenerate. The aim, if possible, is to find an explicit gauge function so that the corresponding generalized Minkowski contents or Hausdorff measure of A be nondegenerate.

Methods of Fractal Analysis in Dynamics s0030

Thermodynamic Formalism s0035

Thermodynamic formalism has been developed by Sinai (1972), Ruelle (1973), and Bowen (1975), using methods of statistical mechanics in order to study dynamics and to find dimensions of various fractal sets. We first describe a "dictionary" for explicit geometric constructions of Cantor-like sets. Let X_p be the set of all sequences $\boldsymbol{i} = (i_1, i_2, \dots)$ of elements i_k from a given set of p symbols, say $\{1, 2, \dots, p\}$. We endow X_p with the metric $d(\boldsymbol{i}, \boldsymbol{j}) := \sum_k 2^{-k}|i_k - j_k|$ and introduce the one-sided shift operator (or left shift) $\sigma\colon X_p \to X_p$ defined by $(\sigma(\boldsymbol{i}))_n = i_{n+1}$, that is, $\sigma(i_1, i_2, i_3, \dots) = (i_2, i_3, i_4, \dots)$. A set $Q \subseteq X_p$ is called the symbolic dynamics if it is compact and σ-invariant, that is, $\sigma(Q) \subseteq Q$. Hence, (Q, σ) is a symbolic dynamical system. Denote $\boldsymbol{i}[n] := (i_1, \dots, i_n)$. Given a continuous function $\varphi\colon Q \to \mathbb{R}$, let us define the topological pressure of φ with respect to σ by p0040

$$P(\varphi) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\{\boldsymbol{i}[n]:\, \boldsymbol{i} \in Q\}} E(\boldsymbol{i}[n])$$

$$E(\boldsymbol{i}[n]) := \exp\left(\sup_{\{\boldsymbol{j} \in Q:\ \boldsymbol{j}[n] = \boldsymbol{i}[n]\}} \sum_{k=0}^{n-1} \varphi(\sigma^k(\boldsymbol{j})) \right)$$

The topological entropy of $\sigma|Q$ is defined by $h(\sigma|Q) := P(0)$, that is,

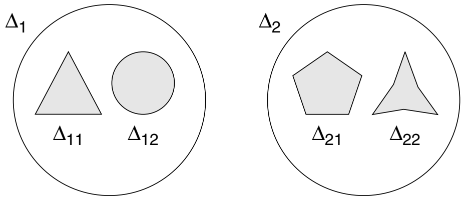
$$h(\sigma|Q) = \lim_{n \to \infty} \frac{1}{n} \log \#\{\boldsymbol{i}[n]\colon\ \boldsymbol{i} \in Q\}$$

where # denotes the cardinal number of a set. The above function $\varphi_n := \sum_{k=0}^{n-1} \varphi \circ \sigma^k$ has the property $\varphi_{n+m} = \varphi_n + \varphi_m \circ \sigma^n$, and therefore we speak about additive thermodynamic formalism. Topological pressure was introduced by D Ruelle (1973) and extended by P Walters (1976). Bowen's equation (1979) has a AU5 very important role in the computation of the Hausdorff dimension of various sets. For the unknown $s \in \mathbb{R}$, and with a suitably chosen function φ, this equation reads

$$P(s\varphi) = 0$$

Geometric Constructions s0040

A geometric construction (Q, Δ) in $\mathbb{R}^m$ indexed by symbolic dynamics Q is a family Δ of compact sets $\Delta_{\boldsymbol{i}[n]} \subset \mathbb{R}^m, \boldsymbol{i} \in Q, n \in \mathbb{N}$, such that $\mathrm{diam}\Delta_{\boldsymbol{i}[n]} \to 0$ as $n \to \infty$, $\Delta_{\boldsymbol{i}[n+1]} \subseteq \Delta_{\boldsymbol{i}[n]}$, $\Delta_{\boldsymbol{i}[n]} = \overline{\mathrm{int}\Delta_{\boldsymbol{i}[n]}}$ for every $\boldsymbol{i} \in Q$ and all n, and $\mathrm{int}\Delta_{\boldsymbol{i}[n]} \cap \mathrm{int}\Delta_{\boldsymbol{j}[n]} = \emptyset$ whenever $\boldsymbol{i}[n] \neq \boldsymbol{j}[n]$ (Moran's open set condition). This family induces the Cantor-like set p0045



f0010 **Figure 2** Cantor-like set.

$$F := \bigcap_{n=1}^{\infty}\left(\bigcup_{i\in Q}\Delta_{i[n]}\right)$$

(see **Figure 2**). The mapping $h : Q \to F$ defined by $h(\boldsymbol{i}) := \cap_{n=1}^{\infty}\Delta_{i[n]}$ is called the coding map of F. The above geometric construction includes well-known iterated function systems of similarities as a special case. If $\lambda_1, \ldots, \lambda_p$ are given numbers in $(0, 1)$, and $\Delta_{i[n]}$ are balls of radii $r_{i[n]} := \lambda_{i_1}\ldots\lambda_{i_n}$, then $s := \dim_H F$ is the unique solution of Bowen's equation $P(s\varphi) = 0$, where φ is defined by $\varphi(\boldsymbol{i}) := \log\lambda_{i_1}$ (Ya Pesin and H Weiss, 1996). In this case Bowen's equation is equivalent to Moran's equation (1946),

$$\sum_{k=1}^{p}\lambda_k{}^{s} = 1$$

This result has been generalized by L Barreira (1996) using the Carathéodory–Pesin construction (1988). Let us illustrate Barreira's theory of nonadditive thermodynamic formalism with a special case. Assume that (Q, Δ) is a geometric construction for which the sets $\Delta_{i[n]}$ are balls, and let there exist $\delta > 0$ such that $r_{i[n+1]} \geq \delta \cdot r_{i[n]}$ and $r_{i[n+m]} \leq r_{i[n]}r_{\sigma^n(i)[m]}$ for all $\boldsymbol{i} \in Q, n, m \in \mathbb{N}$. Then $\dim_H F = \dim_B F = s$, where s is the unique real number such that

$$\lim_{n\to\infty}\frac{1}{n}\log\sum_{\{i[n]:i\in Q\}} r_{i[n]}{}^{s} = 0 \qquad [1]$$

This is a special case of Barreira's extension of Bowen's equation to nonadditive thermodynamic formalism. Moran's equation can be deduced from [1] by defining $r_{i[n]} := \lambda_{i_1}\ldots\lambda_{i_n}$, where $\boldsymbol{i} = (i_1, i_2, \ldots)$, and $\lambda_1, \ldots, \lambda_p \in (0, 1)$ are given numbers. Pesin and Weiss (1996) showed that Moran's open set condition can be weakened so that partial intersections of interiors of pairs of basic sets in the family Δ are allowed. Thermodynamic formalism has been used to study the Hausdorff dimension of Julia sets (Ruelle, 1982), horseshoes (H McCluskey and A Manning, 1983), etc.

p0050 An important example of symbolic dynamics is the topological Markov chain X_A generated by a $p \times p$ matrix A with entries $a_{ij} \in \{0, 1\}$:

$$X_A := \{\boldsymbol{i} = (i_1, i_2, \ldots) \in X_p\text{:}\ \ a_{i_k i_{k+1}} = 1 \text{ for all } k \in \mathbb{N}\}$$

It is a compact, σ-invariant subset of X_p. The map $\sigma\,|\,X_A$ is called the subshift of finite type (Bowen, 1975). A construction of Cantor-like set F using dynamics $Q = X_p$ is called a simple geometric construction, while a geometric construction is said to be a Markov geometric construction if $Q = X_A$. If F is obtained by a Markov geometric construction such that all $\Delta_{i[n]}$ are balls of radii $r_{i[n]} := \lambda_{i_1}\ldots\lambda_{i_n}$, where $\lambda_{i_j} \in (0, 1), i_j \in \{1, \ldots p\}$, then $\dim_B F = \dim_H F = s$, where s is the unique solution of equation $\rho(AM_s) = 1$. Here $M_s := \operatorname{diag}(\lambda_1{}^{s}, \ldots, \lambda_p{}^{s})$ and $\rho(AM_s)$ is the spectral radius of the matrix AM_s. This and more general results have been obtained by Pesin and Weiss (1996).

Any Cantor-like set F obtained via iterated function system of similarities satisfying Moran's open set condition is Hausdorff nondegenerate (Moran, 1946). If F is of nonlattice type, that is, the set $\{\log\lambda_1, \ldots, \log\lambda_p\}$ is not contained in $r \cdot \mathbb{Z}$ for any $r > 0$, then F is Minkowski measurable (D Gatzouras, 1999). p0055

Hyperbolic Measures s0045

Let X be a complete metric space and assume that $f : X \to X$ is continuous. Let μ be an f-invariant Borel probability measure on X (i.e., $\mu(f^{-1}(A)) = \mu(A)$ for measurable sets A) with a compact support. The Hausdorff dimension of μ, and the lower and upper box dimensions of μ (L-S Young 1982) are defined by p0060

$$\dim_H\mu := \inf\{\dim_H Z : Z \subseteq X, \mu(Z) = 1\}$$
$$\underline{\dim}_B\mu := \lim_{\delta\to 0}\inf\{\underline{\dim}_B Z : Z \subseteq X, \mu(Z) \geq 1 - \delta\}$$
$$\overline{\dim}_B\mu := \lim_{\delta\to 0}\inf\{\overline{\dim}_B Z : Z \subseteq X, \mu(Z) \geq 1 - \delta\}$$

It is natural to introduce the lower and upper pointwise dimensions of μ at $x \in X$ by

$$\underline{d}_\mu(x) := \underline{\lim}_{r\to 0}\frac{\log\mu(B_r(x))}{\log r}$$

and similarly $\overline{d}_\mu(x)$. It has been shown by Young (1982) that if X has finite topological dimension and if μ is exact dimensional, that is, $\underline{d}_\mu(x) = \overline{d}_\mu(x) =: d$ for μ-a.e. $x \in X$, then

$$\dim_H\mu = \dim_B\mu = d$$

She also proved that hyperbolic measures (ergodic measures with nonzero Lyapunov exponents), invariant under a $C^{1+\alpha}$-diffeomorphism, $\alpha > 0$, are exact dimensional. F Ledrappier (1986) derived exact dimensionality for hyperbolic Bowen–Ruelle–Sinai measures. This result was extended by Ya Pesin and Ch Yue (1996) to hyperbolic measures with semilocal product structure. J-P Eckmann and D Ruelle (1985) conjectured that the exact dimensionality holds for general hyperbolic measures, and this was proved by Barreira, Pesin, and Schmeling (1996). More precisely, if f is a $C^{1+\alpha}$-diffeomorphism on a smooth Riemann manifold X without boundary, and if μ is f-invariant, compactly supported Borel probability measure, then its hyperbolicity implies that

$$\underline{d}_\mu(x) = \overline{d}_\mu(x) = d^s_\mu(x) + d^u_\mu(x)$$

for μ-a.e. $x \in X$, where $d^s_\mu(x)$ and $d^u_\mu(x)$ are stable and unstable pointwise dimensions of μ at x introduced by Ledrappier and Young (1985).

s0050 Multifractal Analysis of Functions and Measures

p0065 Invariant sets of many dynamical systems are not self-similar. Roughly speaking, the aim of multifractal analysis is to make a decomposition of the invariant set with respect to desired fractal properties and then to study a fractal dimension of each set of the decomposition. Some dynamical systems have invariant sets equal to graphs of Hölderian functions $f : \mathbb{R}^N \to \mathbb{R}$, so that wavelet methods can be used. One of the goals of multifractal analysis of functions is to study the spectrum of singularities of f defined by

$$d_f(\alpha) := \dim_{\mathrm{H}} H_\alpha(f)$$

introduced by U Frisch and G Parisi (1985) in the context of fully developed turbulence. Here $H_\alpha(f)$ is the set of points at which the corresponding pointwise Hölder exponent of f is equal to $\alpha \geq 0$. If the function f is self-similar then $d_f(\alpha)$ is real analytic and strictly concave (first increasing and then decreasing) on an explicit interval $(\underline{a}, \overline{a})$ (S Jaffard, 1997). It is natural to consider the set $C_{\alpha,\beta}(f)$ of points x_0 called chirps of order (α, β) (Y Meyer 1996), at which f behaves roughly like $|x - x_0|^\alpha \sin(1/|x - x_0|^\beta), \beta > 0$. The function $D_f(\alpha, \beta) := \dim_H C_{\alpha,\beta}(f)$ is called the chirp spectrum of f (S Jaffard 2000). Wavelet methods have found applications in the study of evolution equations and in modeling and detection of chirps in turbulent flows (S Jaffard, Y Meyer, RD Robert 2001).

p0070 Basic ideas of multifractal analysis have been introduced by physicists T Halsey, MH Jensen, LP Kadanoff, I Procaccia, and BI Shraiman (1988). In applications it often deals with an invariant ergodic probability measure associated with the dynamical system considered. Multifractal analysis of a Borel finite measure μ defined on $\mathbb{R}^N$ consists in the study of the function

$$d_\mu(\alpha) := \dim_{\mathrm{H}} K_\alpha(\mu), \quad \alpha \geq 0$$

called the spectrum of pointwise dimensions of μ. Here $K_\alpha(\mu)$ is the set of points where the pointwise dimension of μ is equal to α:

$$K_\alpha(\mu) := \{x \in \mathbb{R}^N: \ \underline{d}_\mu(x) = \overline{d}_\mu(x) = \alpha\}$$

It is also of interest to study the Hausdorff dimension of irregular set $K(\mu) := \{x \in \mathbb{R}^N: \underline{d}_\mu(x) < \overline{d}_\mu(x)\}$. These sets are pairwise disjoint and constitute a multifractal decomposition of $\mathbb{R}^N$, that is,

$$\mathbb{R}^N = K(\mu) \cup (\cup_{\alpha \in \mathbb{R}} K_\alpha(\mu))$$

The function $d_\mu(\alpha)$ provides an important information about the complexity of multifractal decomposition. In many situations, there is an open interval $(\underline{\alpha}, \overline{\alpha})$ on which the function $d_\mu(\alpha)$ is analytic and strictly concave (first increasing and then decreasing), and equal to the Legendre transform of an explicit convex function. We thus obtain an uncountable family of sets $K_\alpha(\mu)$ with positive Hausdorff dimension, which shows enormous complexity of the multifractal decomposition of $\mathbb{R}^N$. These and related questions have been studied by L Olsen (1995), K Falconer (1996), Pesin and Weiss (1996), Barreira and Schmeling (2000), and many other authors.

Local Lyapunov Dimension s0055

p0075 Let Ω be an open set in $\mathbb{R}^N$ and let $f : \Omega \to \mathbb{R}^N$ be a C^1-map. To any fixed $x \in \Omega$ we assign N singular values $a_1 \geq a_2 \geq \cdots \geq a_N \geq 0$ of f, defined as square roots of eigenvalues of the matrix $f'(x)^\top \cdot f'(x)$, where $f'(x)$ is the Jacobian of f at x, and $f'(x)^\top$ its transpose. The local Lyapunov dimension of f at x is defined by

$$\dim_{\mathrm{L}}(f, x) := j + s$$

where j is the largest integer in $[0, N]$ such that $a_1 \cdots a_j \geq 1$ (if there is no such j we let $j = 0$), and $s \in [0, 1)$ is the unique solution of $a_1 \cdots a_j a^s_{j+1} = 1$ (except for $j = N$, when we define $s = 0$). This definition, due to BR Hunt (1996), is close to that of Kaplan and Yorke (1979). The Jacobian $f'(x)$ contracts k-dimensional volumes (that is, $a_1 \cdots a_k < 1$) if and only if $\dim_{\mathrm{L}}(f, x) < k$. In this case, we say that f is k-contracting at x. Furthermore, the function $x \mapsto \dim_{\mathrm{L}}(f, x)$ is upper-semicontinuous, so that for any compact subset A of Ω the Lyapunov dimension of f on A, AU6

$$\dim_{\mathrm{L}}(f, A) := \max_{x \in A} \dim_{\mathrm{L}}(f, x)$$

is well defined. Yu S Ilyashenko conjectured that if f locally contracts k-dimensional volumes then the upper box dimension of any compact invariant set is $<k$. Hunt (1996) proved that if A is a compact, strictly invariant set of f (i.e., $f(A)=A$) then

$$\overline{\dim}_{\mathrm{B}} A \leq \dim_{\mathrm{L}}(f, A) \qquad [2]$$

This is an improvement of $\dim_H A \leq \dim_{\mathrm{L}}(f, A)$ obtained by A Douady and J Oesterlé (1980), and independently by Ilyashenko (1982). MA Blinchevskaya and Yu S Ilyashenko (1999) proved that if A is any attractor of a smooth map in a Hilbert space that contracts k-dimensional volumes then $\overline{\dim}_{\mathrm{B}} A \leq k$. See [3] below.

p0080 A continuous variant of this method is used in order to obtain estimates of fractal dimensions of global attractors of dynamical systems (X, S) on a Hilbert space X. Here $S(t), t \geq 0$, is a semigroup of continuous operators on X, that is, $S(t+s) = S(t)S(s)$ and $S(0) = I$. A set A in X is called a global attractor of dynamical system if it is compact, attracting (i.e., for any bounded set B and $\varepsilon > 0$ we have $S(t)B \subseteq A_\varepsilon$), and A is strictly invariant (i.e., $S(t)A = A$ for all $t \geq 0$).

s0060 Applications in Dynamics

s0065 Logistic Map

p0085 M Feigenbaum, a mathematical physicist, introduced and studied the dynamics of the logistic map $f_\lambda : [0,1] \to [0,1], f_\lambda(x) := \lambda x(1-x), \lambda \in (0,4]$. Taking $\lambda = \lambda_\infty \approx 3.570$ the corresponding invariant set $A \subset [0,1]$ (i.e., $S_1(A) \cup S_2(A) = A$, where S_i are two branches of f_λ^{-1}) has both Hausdorff and box dimensions equal to ≈ 0.538 (P Grassberger 1981, P Grassberger and I Procaccia, 1983). The set A has Cantor-like structure, but is not self-similar. Its multifractal properties have been studied by U Frisch, K Khanin, and T Matsumoto (2004).

s0070 Smale Horseshoe

p0090 In the early 1960s S Smale defined his famous horseshoe map and showed that it has a strange invariant set resulting in chaotic dynamics. The notion of strange attractor was introduced in 1971 by Ruelle and Takens in their study of turbulence. Let S be a square in the plane and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a map transforming S as indicated in **Figure 3**, such that on both components of $S \cap f^{-1}(S)$ the map f is affine and preserves both horizontal and vertical directions, and such that points 1, 2, 3, and 4 are mapped to $1', 2', 3'$, and $4'$. Iterating f we get backward invariant set $\Lambda_- := \cap_{j=0}^{\infty} f^{-j}(S)$, forward invariant set $\Lambda_+ := \cap_{j=0}^{\infty} f^{j}(S)$, and invariant set (horseshoe) $\Lambda_f := \Lambda_+ \cap \Lambda_-$. These sets have the Cantor set structure. More precisely, assuming that the contraction parameter of f in vertical direction is $a \in (0, 1/2)$, and the expansion parameter in horizontal direction is $b > 2$, then $\Lambda_+ = [0,1] \times \mathrm{C}^{(a)}$, where $\mathrm{C}^{(a)}$ is the Cantor set, $\Lambda_- = \mathrm{C}^{(1/b)} \times [0,1]$, and $\Lambda_f = \mathrm{C}^{(1/b)} \times \mathrm{C}^{(a)}$, so that $\dim_{\mathrm{B}} \Lambda_+ = \dim_{\mathrm{H}} \Lambda_+ = 1 + (\log 2)/(\log(1/a))$ and

$$\dim_{\mathrm{B}} \Lambda_f = \dim_{\mathrm{H}} \Lambda_f = \frac{\log 2}{\log b} + \frac{\log 2}{\log(1/a)}$$

This is a special case of a general result about horseshoes in $\mathbb{R}^2$ (not necessarily affine), due to McCluskey and Manning (1983), stated in terms of the pressure function. Analogous result as above can be obtained for Smale solenoids. In $\mathbb{R}^3$ it is possible to construct affine horseshoes Λ_f such that $\dim_{\mathrm{H}} \Lambda_f < \dim_{\mathrm{B}} \Lambda_f$ (M Pollicott and H Weiss, 1994).

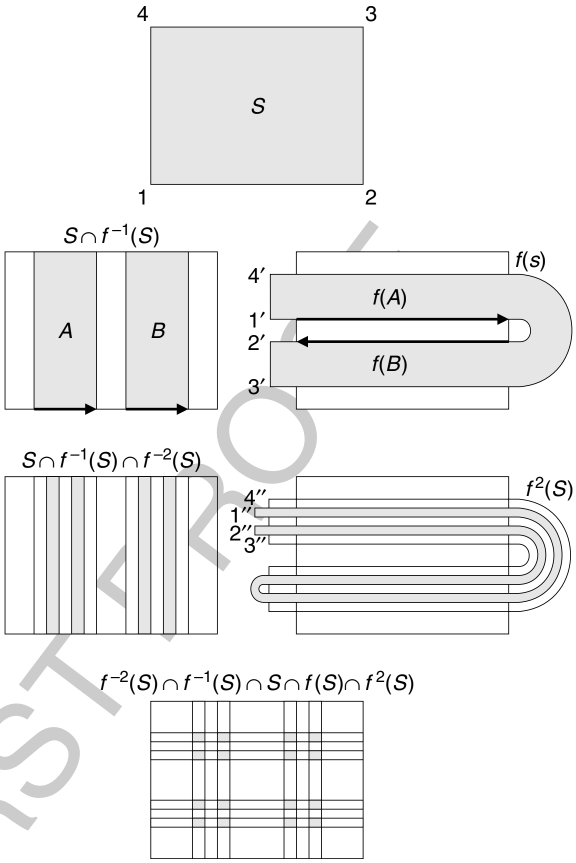


Figure 3 The Smale horseshoe. f0015

p0095 Smale discovered a connection between homoclinic orbits and the horseshoe map. It has been noticed that fractal dimensions have important role in the study of homoclinic bifurcations of nonconservative dynamical systems. Since the 1970s the relationship between invariants of hyperbolic sets and the typical dynamics appearing in the unfolding of a homoclinic tangency by a parametrized family of surface diffeomorphisms has been studied by J Newhouse, J Palis, F Takens, J-C Yoccoz, CG Moreira and M Viana. The main result is that if the Hausdorff dimension of the hyperbolic set involved in the tangency is <1 then the parameter set where the hyperbolicity prevails has full Lebesgue density. If the Hausdorff dimension is >1, then hyperbolicity is not prevalent. This result and its proof were inspired by previous work of JM Marstrand (1954) about arithmetic differences of Cantor sets on the real line. According to the result by Moreira, Palis, and Viana (2001) the paradigm "hyperbolicity prevails if and only if the Hausdorff dimension is <1" extends to homoclinic bifurcations in any dimension.

p0100 Using methods of thermodynamic formalism McCluskey and Manning (1983) proved that if f is the above horseshoe map, then there exists a C^1-neighborhood U of f such that the mapping $f \mapsto \dim_{\mathrm{H}} \Lambda_f$ is continuous. Continuity of box and Hausdorff dimensions for horseshoes has been studied also by Takens, Palis, and Viana (1988).

s0075 Lorenz Attractor

p0105 EN Lorenz (1963), a meteorologist and student of G Birkhoff, showed by numerical experiments that for certain values of positive parameters σ, r, b, the quadratic system

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = rx - y - xz, \qquad \dot{z} = xy - bz$$

has the global attractor A, for example, for $\sigma = 10$, $r = 28, b = 8/3$. In this case $\dim_{\mathrm{B}} A \approx 2.06$, which is a numerical result (Grassberger and Procaccia, 1983). Using the analysis of local Lyapunov dimension along the flow in A, GA Leonov (2001) showed that if $\sigma + 1 \geq b \geq 2$ and $r\sigma^2(4 - b) + 2\sigma(b - 1) \times (2\sigma - 3b) > b(b - 1)^2$ then

$$\overline{\dim}_{\mathrm{B}} A \leq 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma - 1)^2 + 4r\sigma}}$$

s0080 Hénon Attractor

p0110 M Hénon (1976), a theoretical astronomer, discovered the map $f : \mathbb{R}^2 \to \mathbb{R}^2, f(x, y) := (a + by - x^2, x)$, capturing several essential properties of the Lorenz system. In the case of $a = 1.4$ and $b = 0.3$, Hunt (1996) derived from [2] that for any compact, strictly f-invariant set A in the trapping region $[-1.8, 1.8]^2$ there holds $\overline{\dim}_{\mathrm{B}} A < 1.5$. Numerical experiments show that $\dim_{\mathrm{B}} A \approx 1.28$ (Grassberger, 1983). Assuming $a > 0, b \in (0, 1)$, and $P_\pm(x_\pm, x_\pm) \in A$, where $P_\pm$ are fixed points of f, Leonov (2001) obtained that

$$\overline{\dim}_{\mathrm{B}} A \leq 1 + \frac{1}{1 - \ln b / \ln\left(\sqrt{x_-^2 + b} - x_-\right)}$$

Here

$$x_\pm := \frac{1}{2}\left[b - 1 \pm \sqrt{(b - 1)^2 + 4a}\right]$$

The proof is based on the study of local Lyapunov dimension of f and its iterates on A.

Embedology s0085

The physical relevance of box dimensions in the study of attractors is related to the problem of finding the smallest possible dimension n sufficient to "embed" an attractor into $\mathbb{R}^n$. If $A \subset \mathbb{R}^k$ is a compact set and if $n > 2\underline{\dim}_{\mathrm{B}} A$, then almost every map from $\mathbb{R}^k$ into $\mathbb{R}^n$, in the sense of prevalence, is one-to-one on A and, moreover, it is an embedding on smooth manifolds contained in A (T Sauer, JA Yorke, and M Casdagli, 1991). If A is a strange attractor then the same is true for almost every delay-coordinate map from $\mathbb{R}^k$ to $\mathbb{R}^n$. This improves an earlier result by H Whitney (1936) and F Takens (Takens' embedology, 1981). The above notion of prevalence means the following: a property holds almost everywhere in the sense of prevalence if it holds on a subset S of the space $V := C^1(\mathbb{R}^k, \mathbb{R}^n)$ for which there exists a finite-dimensional subspace $E \subset V$ (probe space) such that for each $v \in V$ we have that $v + e \in S$ for Lebesgue a.e. $e \in E$. p0115

Julia and Mandelbrot Sets s0090

M Shishikura (1998) proved that the boundary of the Mandelbrot set M generated by $f_c(z) := z^2 + c$ has the Hausdorff dimension equal to 2, thus answering positively to the conjecture by B Mandelbrot, J Milnor, and other mathematicians. Also for Julia sets there holds $\dim_{\mathrm{H}} J(f_c) = 2$ for generic c in M (i.e., on the set of second Baire category). The proof is based on the study of the bifurcation of parabolic periodic points. Also, each baby Mandelbrot set sitting inside of M has the boundary of Hausdorff dimension 2 (L Tan, 1998). Shishikura's results hold for more general functions $f(z) := z^d + c$, where $d \geq 2$. p0120

For Julia sets $J(f_c)$ generated by $f_c(z) := z^2 + c$ there holds $d(c) := \dim_{\mathrm{H}} J(f_c) = 1 + |c|^2/(4 \log 2) + o(|c|^2)$ for $c \to 0$. This and more general results have been obtained by Ruelle (1982). He also proved that the function $d(c)$ when restricted to the interval $[0, \infty)$ is real analytic in $[0, 1/4) \cup (1/4, \infty)$. Furthermore, it is left continuous at 1/4 (O Bodart and M Zinsmeister, 1996), but not continuous (A Douady, P Sentenac, and M Zinsmeister, 1997). p0125

Discontinuity of this map is related to the phenomenon of parabolic implosion at $c=1/4$. The derivative $d'(c)$ tends to $+\infty$ from the left at $c=1/4$ like $(1/4-c)^{d(1/4)-3/2}$ (G Havard and M Zinsmeister, 2000). Here $d(1/4)\approx 1.07$, which is a numerical result. Analysis of dimensions is based on methods of thermodynamic formalism.

p0130 C McMullen (1998) showed that if θ is an irrational number of bounded type (i.e., its continued fractional expansion $[a_1, a_2, \dots]$ is such that the sequence (a_i) is bounded from above) and $f(z) := z^2 + e^{2\pi\theta i}z$, then the Julia set $J(f)$ is porous. In particular, $\overline{\dim}_B J(f) < 2$. YC Yin (2000) showed that if all critical points in $J(f)$ of a rational map $f: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ are nonrecurrent (a point is nonrecurrent if it is not contained in its ω-limit set) then $J(f)$ is porous, hence $\overline{\dim}_B J(f) < 2$. Urbański and Przytycki (2001) described more general rational maps such that $\overline{\dim}_B J(f) < 2$.

s0095 Spiral Trajectories

p0135 A standard planar model where the Hopf–Takens bifurcation occurs is $\dot{r} = r(r^{2l} + \sum_{i=0}^{l-1} a_i r^{2i}), \dot{\varphi} = 1$, where $l \in \mathbb{N}$. If Γ is a spiral tending to the limit cycle $r = a$ of multiplicity m (i.e., $r = a$ is a zero of order m of the right-hand side of the first equation in the system) then $\dim_B \Gamma = 2 - 1/m$. Furthermore, for $m > 1$ the spiral is Minkowski measurable (Žubrinić and Županović, 2005). For $m = 1$ the spiral is Minkowski nondegenerate with respect to the gauge function $h(\varepsilon) := \varepsilon(\log(1/\varepsilon))^{-1}$.

s0100 Infinite-Dimensional Dynamical Systems

p0140 In many situations the dynamics of the global attractor A of the flow corresponding to an autonomous Navier–Stokes system is finite-dimensional (Ladyzhenskaya, 1972). This means that there exists a positive integer N such that any trajectory in A is completely determined by its orthogonal projection onto an N-dimensional subspace of a Hilbert space X. The aim is to find estimates of box and Hausdorff dimensions of the global attractor, in order to understand some of the basic and challenging problems of turbulence theory. If A is a subset of a Hilbert space X, its Hausdorff dimension is defined analogously as for $A \subset \mathbb{R}^N$. The definition of the upper box dimension can be extended from $A \subset \mathbb{R}^N$ to

$$\overline{\dim}_B A := \overline{\lim}_{\varepsilon \to 0} \frac{\log m(A, \varepsilon)}{\log(1/\varepsilon)} \qquad [3]$$

where $m(A, \varepsilon)$ is the minimal number of balls sufficient to cover a given compact set $A \subset X$. The value of $\log m(A, \varepsilon)$ is called ε-entropy of A.

p0145 Foiaş and Temam (1979), Ladyzhenskaya (1982), AV Babin and MI Vishik (1982), Ruelle (1983), and E Lieb (1984) were among the first who obtained explicit upper bounds of Hausdorff and box dimensions of attractors of infinite-dimensional systems. For global attractors A associated with some classes of two-dimensional Navier–Stokes equations with nonhomogeneous boundary conditions it can be shown that $\overline{\dim}_B A \le c_1 G + c_2 Re^{3/2}$, where G is the Grashof number, Re is the Reynolds number, and c_i are positive constants (RM Brown, PA Perry, and Z Shen, 2000). VV Chepyzhov and AA Ilyin (2004) obtained that $\overline{\dim}_B A \le (1/\sqrt{2}\pi)(\lambda_1|\Omega|)^{1/2} G$ for equations with homogeneous boundary conditions, where $\Omega \subset \mathbb{R}^2$ is a bounded domain, and λ_1 is the first eigenvalue of $-\Delta$. In the case of periodic boundary conditions Constantin, Foiaş, and Temam (1988) proved that $\overline{\dim}_B A \le c_1 G^{2/3}(1 + \log G)^{1/3}$, while for a special class of external forces there holds $\dim_H A \ge c_2 G^{2/3}$ (VX Liu, 1993). Let us mention an open problem by VI Arnol'd: is it true that the Hausdorff dimension of any attracting set of the Navier–Stokes equation on two-dimensional torus is growing with the Reynolds number?

p0150 In their study of partial regularity of solutions of three-dimensional Navier–Stokes equations, L Caffarelli, R Kohn, and L Nirenberg (1982) proved that the one-dimensional Hausdorff measure in space and time (defined by parabolic cylinders) of the singular set of any "suitable" weak solution is equal to zero. A weak solution is said to be singular at a point (x_0, t_0) if it is essentially unbounded in any of its neighborhoods. Dimensions of attractors of many other classes of partial differential equations (PDEs) have been studied, like for reaction–diffusion systems, wave equations with dissipation, complex Ginzburg–Landau equations, etc. Related questions for nonautonomous PDEs have been considered by VV Chepyzhov and MI Vishik since 1992.

Probability s0105

p0155 Important examples of trajectories appearing in physics are provided by Brownian motions. Brownian motions ω in $\mathbb{R}^N, N \ge 2$, have paths $\omega([0,1])$ of Hausdorff dimension 2 with probability 1, and they are almost surely Hausdorff degenerate, since $\mathcal{H}^2(\omega([0,1])) = 0$ for a.e. ω (SJ Taylor, 1953). Defining gauge functions $h(\varepsilon) := \varepsilon^2 \log(1/\varepsilon) \times \log\log\log(1/\varepsilon)$ when $N = 2$, and $h(\varepsilon) := \varepsilon^2 \log(1/\varepsilon)$ when $N \ge 3$, there holds $\mathcal{H}^h(\omega([0,1])) \in (0, \infty)$ for a.e. ω (D Ray, 1963, SJ Taylor, 1964). If $N = 1$ then a.e. ω has the box and Hausdorff dimensions of the graph of $\omega|_{[0,1]}$ equal to 3/2 (Taylor, 1953), and for the gauge function $h(\varepsilon) := \varepsilon^{3/2} \log\log(1/\varepsilon)$ the corresponding generalized Hausdorff measure is nondegenerate. In the case of $N \ge 2$ we have the uniform dimension doubling property (R Kaufman, 1969). This means that for a.e. Brownian motion ω there holds $\dim_{\mathrm{H}}\omega(A) = 2\dim_{\mathrm{H}}A$ for all subsets $A \subset [0, \infty)$. There are also results concerning almost sure Hausdorff dimension of double, triple, and multiple points of a Brownian motion and of more general Lévy stable processes.

p0160 Fractal dimensions also appear in the study of stochastic differential equations, like

$$dx_t = X_0(x_t)\,dt + \sum_{k=1}^{d} X_k(x_t)\,d\theta_k(t), \quad x_0 = x \in \mathbb{R}^N$$

The stochastic flow $(x_t)_{t\geq 0}$ in $\mathbb{R}^N$ is driven by a Brownian motion $(\theta(t))_{t\geq 0}$ in $\mathbb{R}^d$. Let us assume that $X_k, k = 0, \ldots, d$, are C^∞-smooth T-periodic divergence-free vector fields on $\mathbb{R}^N$. Then for almost every realization of the Brownian motion $(\theta(t))_{t\geq 0}$, the set of initial points x generating the flow $(x_t)_{t\geq 0}$ with linear escape to infinity (i.e., $\underline{\lim}_{t\to\infty}(|x_t|/t) > 0$) is dense and of full Hausdorff dimension N (D Dolgopyat, V Kaloshin, and L Koralov, 2002).

s0110 Other Directions

p0165 There are many other fractal dimensions important for dynamics, like the Rényi spectrum for dimensions, correlation dimension, information dimension, Hentschel–Procaccia spectrum for dimensions, packing dimension, and effective fractal dimension. Relations between dimension, entropy, Lyapunov exponents, Gibbs measures, and multifractal rigidity have been investigated by Pesin, Weiss, Barreira, Schmeling, etc. Fractal dimensions are used to study dynamics appearing in Kleinian groups (D Sullivan, CJ Bishop, PW Jones, C McMullen, BO Stratmann, etc.), quasiconformal mappings and quasiconformal groups (FW Gehring, J Väisâla, K Astala, CJ Bishop, P Tukia, JW Anderson, P Bonfert-Taylor, EC Taylor, etc.), graph directed Markov systems (RD Mauldin, M Urbański, etc.), random walks on fractal graphs (J Kigami, A Telcs, etc.), billiards (H Masur, Y Cheung, P Bálint, S Tabachnikov, N Chernov, D Szász, IP Tóth, etc.), quantum dynamics (J-M Barbaroux, J-M Combes, H Schulz-Baldes, I Guarneri, etc.), quantum gravity (M Aizenman, A Aharony, ME Cates, TA Witten, GF Lawler, B Duplantier, etc.), harmonic analysis (RS Strichartz, ZM Balogh, JT Tyson, etc.), number theory (L Barreira, M Pollicott, H Weiss, B Stratmann, B Saussol, etc.), Markov processes (RM Blumenthal, R Getoor, SJ Taylor, S Jaffard, C Tricot, Y Peres, Y Xiao, etc.), and theoretical computer science (B Ya Ryabko, L Staiger, JH Lutz, E Mayordomo, etc.), and so on.

See also: Bifurcation of Periodic Orbits (00027); Chaos and Attractors (00093); Dissipative Dynamical Systems of Infinite Dimension (00095); Dynamical Systems in Mathematical Physics (00098); Ergodic Theory (00403); Generic Properties of Dynamical Systems (00164); Holomorphic Dynamics (00404); Homoclinic Phenomena (00374); Hyperbolic Systems (00407); Lyapunov Exponents, Strange Attractors (00100); Polygonal Billiards (00452); Renormalization and the Feigenbaum Phenomenon (00167); Stationary Solutions of PDEs: and Heteroclinic/Homoclinic Connexions of Dynamical Systems (00104); Synchronization of Chaos (00105); Navier-Stokes Turbulence (00206); Partial Differential Equations in Fluid Mechanics (00251); Quantum Chaos (00332); Wavelets: Mathematical Theory (00153); Mathematics of Image Processing (00367); Stochastic Differential Equations (00369).

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